

**Remember:** 1. Show all your work. 2. Make sure that the answer(s) is (are) of the right type. If your answer will be of the wrong type (for example, if the answer is supposed to be an equation of a straight line, and your final answer is  $y = x^2(x - 2) + 3$  (which is **not** an equation of straight line) you would get no points at all, even if everything is correct except for one step.

1. (16 points altogether ) Consider the function

$$f(x) = x^4 - 4x^3 + 4x^2$$

(a) (6 points) Find all the local maxima and local minima

**Sol. to 1(a):**

$$f'(x) = 4x^3 - 12x^2 + 8x \quad , \quad f''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2) \quad .$$

To get the potential max and min, set  $f'(x) = 0$ . To solve

$$4x^3 - 12x^2 + 8x = 0 \quad ,$$

factorize:

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x - 1)(x - 2) \quad .$$

Solving  $4x(x - 1)(x - 2) = 0$  , gives **three** solutions:

$$x = 0 \quad , \quad x = 1 \quad , x = 2 \quad .$$

When  $x = 0$ ,  $f''(0) = 8 > 0$  is **positive**, so  $x = 0$  is a **local min**. Plugging into  $f(x)$ ,  $f(0) = 0^4 - 4 \cdot 0^3 + 4 \cdot 0^2 = 0$ , so the **point**  $(0, 0)$  is a local min.

When  $x = 1$ ,  $f''(1) = -4 < 0$  is **negative**, so  $x = 1$  is a **local max**. Plugging into  $f(1) = 1^4 - 4 \cdot 1^3 + 4 \cdot 1^2 = 1$ , so the **point**  $(1, 1)$  is a local max.

When  $x = 2$ ,  $f''(2) = 4(3 \cdot 2^2 - 6 \cdot 2 + 2) = 8 > 0$  is **positive**, so  $x = 2$  is a **local min**. Plugging into  $f(x)$ ,  $f(2) = 2^4 - 4 \cdot 2^3 + 4 \cdot 2^2 = 0$ , so the **point**  $(2, 0)$  is a local min.

**Ans. to 1(a):** The local minima are  $(0, 0)$  and  $(2, 0)$ . The only local maximum is  $(1, 1)$ .

(b) (2 points) Find the  $x$ -coordinates of all the inflection points (if they exist). (Do not compute the  $y$ -coordinates).

**Sol. to 2(a):** Setting  $f''(x) = 0$  gives the equation

$$4(3x^2 - 6x + 2) = 0$$

Of course, you can forget about the 4. Using

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} ,$$

We get that the roots (solutions) are

$$\frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$

**Ans. to 1b):** The  $x$ -coordinates of the inflections points are  $x = 1 - \frac{\sqrt{3}}{3}$  and  $x = 1 + \frac{\sqrt{3}}{3}$ .

(c) (4 points) In what intervals is the function (i) increasing? (ii) decreasing(?) (iii) concave up? (iv) concave down?

**Sol. to 2(c):** The three roots of  $f'(x) = 0$ , namely  $x = 0, x = 1, x = 2$ , divide the real line into **four** open intervals.  $(-\infty, 0), (0, 1), (1, 2), (2, \infty)$ . Picking a random (easy) number at each, and plugging into  $f'(x)$ , we see that

For the interval  $(-\infty, 0)$ , pick  $x = -1$ .  $f'(-1) = (-4)(-2)(-3) = -24$  which is **negative**. So The function is **decreasing** in the open interval  $(-\infty, 0)$ .

For the interval  $(0, 1)$ , pick  $x = 1/2$ .  $f'(1/2) = 4(1/2)(-1/2)(-3/2) = 3/2$  which is **positive**. So The function is **increasing** in the open interval  $(0, 1)$ .

For the interval  $(1, 2)$ , pick  $x = 3/2$ .  $f'(3/2) = 4(3/2)(1/2)(-1/2) = -3/2$  which is **negative**. So The function is **decreasing** in the open interval  $(1, 2)$ .

For the interval  $(2, \infty)$ , pick  $x = 3$ .  $f'(3) = 4(3)(2)(1) = 24$  which is **positive**. So The function is **increasing** in the open interval  $(2, \infty)$ .

For concave up vs. concave down, the roots of  $f''(x) = 4(3x^2 - 6x + 2) = 0$  divide the real line into **three** open intervals.

$$(-\infty, 1 - \sqrt{3}/3) , \quad (1 - \sqrt{3}/3, 1 + \sqrt{3}/3) , \quad (1 + \sqrt{3}/3, \infty) .$$

For the first of these intervals, pick  $x = -10$ , and get  $f''(-10) > 0$ , so it is **concave up** on  $(-\infty, 1 - \sqrt{3}/3)$ .

For the second of these intervals, pick  $x = 1$ , and get  $f''(1) = -4 < 0$ , so it is **concave down** on  $(1 - \sqrt{3}/3, 1 + \sqrt{3}/3)$ .

For the third of these intervals, pick  $x = 10$ , and get  $f''(10) > 0$ , so it is **concave up** on  $(1 + \sqrt{3}/3, \infty)$ .

**Ans. to 1(c):** (i)  $(0, 1)$  and  $(2, \infty)$ ; (ii)  $(-\infty, 0)$  and  $(1, 2)$ ; (iii)  $(-\infty, 1 - \sqrt{3}/3)$  ,  $(1 + \sqrt{3}/3, \infty)$  ; (iv)  $(1 - \sqrt{3}/3, 1 + \sqrt{3}/3)$ .

**Note:** All the above are **open intervals**, that use the same notation as **points**. Don't confuse the two completely different notions.

(d) (4 points) Sketch the graph

**Verbal description:** It comes down from  $\infty$  from the extreme left, to a “valley” at the origin. Now it goes up to a small hill-top at  $(1, 1)$ , and then again, to another valley at  $(2, 0)$ , and after that it goes up-up-up to infinity for ever after. It looks like an upside down body of a camel (with two humps).

### Extra Problems for Practice for the Final

Do exactly the same, in full detail, for the following functions

$$f(x) = x^3 - x \quad .$$

$$f(x) = x^4 - 16x^2 \quad .$$

2. (a) (4 points) Find

$$\lim_{x \rightarrow 0} \frac{e^{x^2} - 1 - x^2}{x^4}$$

**Solution to 2a):** First you plug-in  $x = 0$ , and get  $0/0$ . So this is game for L'hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{e^{x^2} - 1 - x^2}{x^4} = \lim_{x \rightarrow 0} \frac{(e^{x^2} - 1 - x^2)'}{(x^4)'} = \lim_{x \rightarrow 0} \frac{2xe^{x^2} - 2x}{4x^3} \quad .$$

**Now**, you plug-in  $x = 0$  again, and once again you get  $0/0$ . **BUT** before you go on, **simplify as much as you can**. Otherwise, things will get too complicated.

Simplifying, we get

$$\lim_{x \rightarrow 0} \frac{2xe^{x^2} - 2x}{4x^3} = \lim_{x \rightarrow 0} \frac{2x(e^{x^2} - 1)}{4x^3} = \lim_{x \rightarrow 0} \frac{(e^{x^2} - 1)}{2x^2} \quad .$$

Only now, after, we have used **algebra**, we use L'Hôpital again

$$= \lim_{x \rightarrow 0} \frac{(e^{x^2} - 1)}{2x^2} = \lim_{x \rightarrow 0} \frac{(e^{x^2} - 1)'}{(2x^2)'} = \lim_{x \rightarrow 0} \frac{2xe^{x^2}}{4x}$$

Now **simplify** again!

$$= \lim_{x \rightarrow 0} \frac{e^{x^2}}{2} \quad .$$

Now plug-in  $x = 0$  and get  $e^{0^2}/2 = 1/2$ .

**Ans. to 2a):**  $\frac{1}{2}$ .

### Extra Problems for Practice for the Final

$$\lim_{x \rightarrow 0} \frac{e^{x^3} - 1 - x^3}{x^6}$$

$$\lim_{x \rightarrow 0} \frac{x^4 - 2x^2 + 1}{(x - 1)^2}$$

(b) ( 4 points)

$$\lim_{x \rightarrow \infty} \frac{3x^3 + 2x^2 + 1}{6x^3 + 2x - 11}$$

**Sol. to 2b):** When  $x$  is very very big, only the leading terms count, so by the “forget-about-the-little- ones” rule:

$$\lim_{x \rightarrow \infty} \frac{3x^3 + 2x^2 + 1}{6x^3 + 2x - 11} = \lim_{x \rightarrow \infty} \frac{3x^3}{6x^3} = \lim_{x \rightarrow \infty} \frac{1}{2} = \frac{1}{2} \quad .$$

**Ans. to 2b):**  $\frac{1}{2}$ .

**Note:** Of course, you can also use L'Hôpital.

(c) ( 4 points)

$$\lim_{x \rightarrow 0} \frac{3x^{100} + 2x^2 + 22}{6x^{100} + 2x - 11}$$

**Sol. to 2c):** This is a piece of cake! Plug-in  $x = 0$  and get something that makes sense.

$$\lim_{x \rightarrow 0} \frac{3x^{100} + 2x^2 + 22}{6x^{100} + 2x - 11} = \frac{3 \cdot 0^{100} + 2 \cdot 0^2 + 22}{6 \cdot 0^{100} + 2 \cdot 0 - 11} = \frac{22}{-11} = -2.$$

**Ans. to 2c):**  $-2$ .

**Common mistake:** Many people got confused and read the problem as  $\lim_{x \rightarrow \infty}$ , like 2b). **Read the problem carefully, and don't do a different one!**

### Extra Problems for Practice for the Final

$$\lim_{x \rightarrow \infty} \frac{2x^{100} + 12x^2 + 22}{16x^{100} + 2x - 11}$$

$$\lim_{x \rightarrow 0} \frac{12x^{300} + 2x^2 + 22}{26x^{300} + 2x - 11}$$

$$\lim_{x \rightarrow 1} \frac{3x^{100} + 2x^2 + 22}{x - 2}$$

$$\lim_{x \rightarrow \infty} \frac{4x^{100} + 11}{8x^{100} + 12}$$

3. (12 points) Find the point on the curve

$$y = \frac{2\sqrt{3}}{3}x^{3/2} \quad ,$$

closest to the point  $(1, 0)$ .

**Solution to 3.**

The goal function is **distance-squared**. The distance of a general point  $(x, y)$ , to the specific point  $(1, 0)$  is

$$d^2 = (x - 1)^2 + (y - 0)^2 = (x - 1)^2 + y^2 \quad .$$

The **constraint** is  $y = \frac{2\sqrt{3}}{3}x^{3/2}$ . (We are only considering points that lie on that curve). Plugging-in into  $d^2$ , we get

$$d^2 = (x - 1)^2 + (y - 0)^2 = (x - 1)^2 + \left(\frac{2\sqrt{3}}{3}x^{3/2}\right)^2$$

One again, **SIMPLIFY!**, or else your life will be miserable! Always **simplify before you differentiate** (or integrate, or do anything else, for that matter). Simplifying, we get

$$d^2 = (x - 1)^2 + \left(\frac{2\sqrt{3}}{3}x^{3/2}\right)^2 = x^2 - 2x + 1 + \frac{4}{3}x^3 \quad .$$

**Note:** Here we used the **very important** rule

$$(x^a)^b = x^{ab} \quad ,$$

This rule should be **in your blood**.

So the (simplified) **goal function**, let's call it  $f(x)$ , is:

$$f(x) = \frac{4}{3}x^3 + x^2 - 2x + 1 \quad .$$

This is the function that we have to **minimize**. Taking the derivative, we have

$$f'(x) = \frac{4}{3}3x^2 + 2x - 2 = 4x^2 + 2x - 2 = 2(2x^2 + x - 1) \quad .$$

**Factoring**, we get

$$2(2x^2 + x - 1) = 2(2x - 1)(x + 1) = 0 \quad .$$

Getting the roots (alias solutions),  $x = \frac{1}{2}$  and  $x = -1$ . The second can be discarded, since it gives an imaginary  $y$ -coordinate. So the answer is  $x = \frac{1}{2}$ . To get the full point, we need to plug it into  $y = \frac{2\sqrt{3}}{3}x^{3/2}$ , getting  $y = \frac{2\sqrt{3}}{3}(1/2)^{3/2} = \frac{\sqrt{6}}{6}$ .

**Ans. to 3.** The closest point on the curve to  $(1, 0)$  is the point  $(1/2, \sqrt{6}/6)$ .

### Extra Problems for Practice for the Final

Do the same type of problems for

Curve:  $y = 2x + 3$ , point= $(1, 1)$ .

Curve:  $y = \sqrt{x}$ , point= $(2, 0)$ .

Curve:  $y = 1 - x^2$ , point= $(2, 2)$ . (Warning, just set up the equation, it may be messy or impossible to solve without a calculator).

#### 4. (12 points)

A spotlight on the ground shines on a wall 40 meters away. A girl 1.50 meters tall walks from the spotlight to the wall, at a speed of 0.8 m./sec. ; her path is perpendicular to the wall. Let  $x$  be the distance from her feet to the spotlight and let  $h$  be the height of the shadow on the wall.

(a) Draw a sketch of the problem, and find a formula relating  $h$  and  $x$ .

Draw a right-angled triangle with base 40 and height  $h$ , and a perpendicular segment of height 1.5 jutting out of a point distance  $x$  from the right vertex (where the spotlight is).

By **similar triangles**

$$\frac{h}{40} = \frac{1.5}{x} \quad .$$

Cross-multiplying, we get

$$h = \frac{60}{x} \quad .$$

**Ans. to 4a):**  $h = \frac{60}{x}$ .

**Note:**  $hx = 60$  is also correct, but not as convenient for part b.

(b) When the girl is 4 meters from the wall, find the height of the shadow and the rate of change of the height of the shadow.

**be careful!:**  $x = 40 - 4 = 36$ , **not**  $x = 4$ . Other than that, the first part is easy. Just plug-in  $x = 36$  and get  $h = 60/36 = 5/3$ . For  $\frac{dh}{dt}$ , we have to differentiate with respect to  $t$ , using **implicit differentiation** (or the chain rule).

$$\frac{dh}{dt} = \frac{-60}{x^2} \frac{dx}{dt} \quad .$$

By the data of the problem,  $\frac{dx}{dt} = 0.8$ . Plugging-in  $x = 36$ , we get

$$\frac{dh}{dt} = \frac{-60}{36^2} (0.8) = \frac{-10 \cdot 4}{36 \cdot 6 \cdot 5} = \frac{-1}{9 \cdot 3} = \frac{-1}{27} \quad .$$

**Ans. to 4b):** The height of the shadow is 5/3 meters and the length of the shadow on the wall is decreasing at a rate of 1/27 meters per second.

### Extra Problems for Practice for the Final

Make up another problem **just like that** with **different** numbers, and **solve it**.

Make up a similar, but a **different** problem in which the spotlight is on the wall, and the shadow is on the ground, and **solve it**.

### 5. Differentiate the following functions

(a) (4 points)

$$f(x) = (\sin x) \cdot \ln(\sin^{-1} x)$$

**Sol. to 5a):**

**First** you use the **product rule**:

$$\begin{aligned} f'(x) &= ((\sin x) \cdot \ln(\sin^{-1} x))' = (\sin x)' \cdot \ln(\sin^{-1} x) + (\sin x) \cdot (\ln(\sin^{-1} x))' \\ &= \cos x \ln(\sin^{-1} x) + (\sin x) \cdot (\ln(\sin^{-1} x))' \quad . \end{aligned}$$

**Now** you use the **chain rule** for the second term:

$$= \cos x \ln(\sin^{-1} x) + (\sin x) \cdot \frac{1}{\sin^{-1} x} \cdot (\sin^{-1}(x))'$$

To get  $(\sin^{-1}(x))'$  you look it up, either from the formula sheet, or better still, from your own head, you get that it equals  $\frac{1}{\sqrt{1-x^2}}$ . Combining we have:

$$f'(x) = \cos x \ln(\sin^{-1} x) + (\sin x) \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} = \cos x \ln(\sin^{-1} x) + \frac{\sin x}{\sqrt{1-x^2} \sin^{-1} x} \quad .$$

**Ans. to 5a):**  $\cos x \ln(\sin^{-1} x) + \frac{\sin x}{\sqrt{1-x^2} \sin^{-1} x}$ .

(b) (4 points)

$$f(x) = \log_{10}(\sin^{-1} x)$$

**Sol. to 5b): First**, convert to  $\ln$  by using the **very important** formula

$$\log_b W = \frac{\ln W}{\ln b} \quad .$$

So

$$f(x) = \log_{10}(\sin^{-1} x) = \frac{\ln(\sin^{-1} x)}{\ln 10} \quad .$$

Now differentiate, **remembering** that  $1/\ln 10$  is **just a number!**, so you can take it out of the differentiation. Using the **chain rule**, we have:

$$f'(x) = \left( \frac{\ln(\sin^{-1} x)}{\ln 10} \right)' = \frac{1}{(\ln 10) \sin^{-1} x} \cdot (\sin^{-1} x)' =$$

$$\frac{1}{(\ln 10) \sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{1}{(\ln 10)(\sin^{-1} x)\sqrt{1-x^2}}$$

**Ans. to 5b):**  $\frac{1}{(\ln 10)(\sin^{-1} x)\sqrt{1-x^2}}$ .

(c) (4 points)

$$f(x) = e^{2 \ln \sin x} + \sqrt{\cos^4 x}$$

**Sol. to 5c:** Remember, **Simplify before you differentiate**. It is very important to be fluent and proficient in algebra.

Obviously  $\sqrt{\cos^4 x} = \cos^2 x$ . Regarding the first term, first bring the 2 to the power:  $2 \ln \sin x = \ln \sin^2 x$ , and now  $e^{2 \ln \sin x} = e^{\ln \sin^2 x} = \sin^2 x$ . So the whole thing is

$$f(x) = \sin^2 x + \cos^2 x \quad .$$

But, by the famous **trig. identity**, this is 1, so

$$f(x) = 1$$

and  $f'(x) = 0$ .

**Ans. to 5c:**  $f'(x) = 0$ .

**Comments:** If you don't simplify, and do things correctly, you may still get a "right answer", but it is a very complicated way of expressing 0. Few people did it the long way, got something correct, but didn't realize that it simplifies to 0. However, most people messed up, and made some mistake, since it was so complicated. Again: **Simplify before you differentiate**.

### Extra Problems for Practice for the Final

Differentiate the following functions:

$$f(x) = (\tan x) \cdot \ln(\cos^{-1} x)$$

$$f(x) = e^{x^2} \cdot \ln(\tan x)$$

$$f(x) = \log_5(\tan^{-1} x)$$



$$f(x) = e^{\ln \sqrt{x^2}}$$

$$f(x) = e^{2 \ln \cos x} + (\sin^6 x)^{1/3} \quad .$$

**6.** (12 points) Find the area of the largest rectangle that can be inscribed in the region bounded by the graph

$$y = \frac{8-x}{1+x} \quad ,$$

and the coordinate axes.

**Sol. to 6:** The area of the shaded rectangle is obviously  $xy$ . This is our **goal function**. But we must express it in terms of  $x$  alone. Using the **constraint**

$$y = \frac{8-x}{1+x} \quad ,$$

(That vertex lies on the curve), we get that the area of a general rectangle is:

$$A = x \frac{8-x}{1+x} = \frac{(8-x)x}{1+x} \quad .$$

This is your **goal function**, let's call it  $f(x)$ .

$$f(x) = \frac{(8-x)x}{1+x} \quad .$$

Using the **quotient** rule, we have

$$f'(x) = \frac{(8x - x^2)'(1+x) - (8x - x^2)(1+x)'}{(1+x)^2} = \frac{(8-2x)(1+x) - (8x - x^2)}{(1+x)^2} =$$

$$\frac{(8-2x) + 8x - 2x^2 - 8x + x^2}{(1+x)^2} = -\frac{x^2 + 2x - 8}{(1+x)^2} \quad .$$

We set this equal to 0, which means that we have to set the **top** equal to 0.

$$x^2 + 2x - 8 = 0 \quad .$$

Factorizing (you can also use the quadratic formula, if you wish)

$$(x-2)(x+4) = 0 \quad .$$

The root  $x = -4$  does not make sense (we are only talking about positive  $x$ ), so the only root is  $x = 2$ . When  $x = 2$ , we have

$$y = \frac{8-2}{1+2} = \frac{6}{3} = 2 \quad ,$$

and the largest area is  $2 \times 2 = 4$ .

**Ans. to 6:** The largest area of such a rectangle is 4.

### Extra Problems for Practice for the Final

Repeat the same problem with

$$y = 1 - x^2$$

$$y = 1 - \sqrt{x}$$

**7.** (a) Find the derivative of the function  $f(x) = (3x + 1)^3(x^3 - 4)^7 \sin^{10} x$ .

**Sol. to 7a).** The best way is via **logarithmic differentiation**. Take the  $\ln$ :

$$\ln f(x) = 3 \ln(3x + 1) + 7 \ln(x^3 - 4) + 10 \ln \sin x \quad .$$

Differentiate both sides, we get

$$\frac{f'(x)}{f(x)} = 3 \frac{3}{3x + 1} + 7 \frac{3x^2}{x^3 - 4} + 10 \frac{\cos x}{\sin x} = \frac{9}{3x + 1} + \frac{21x^2}{x^3 - 4} + 10 \cot x$$

Finally, multiplying both sides by  $f(x)$ , we get the final answer:

$$f'(x) = \left( \frac{9}{3x + 1} + \frac{21x^2}{x^3 - 4} + 10 \cot x \right) (3x + 1)^3 (x^3 - 4)^7 \sin^{10} x \quad .$$

This is the **ans..**

(b) (6 points) Solve the differential equation

$$y''(x) = 6x \quad ,$$

with initial condition  $y(1) = 1$ ,  $y'(1) = 3$ .

**Ans. 7b):** To get  $y'(x)$ , we take the anti-derivative of  $6x$ , not forgetting the  $+C$ :

$$y'(x) = \int 6x \, dx = 6 \frac{x^2}{2} = 3x^2 + C \quad .$$

To find the exact value of  $C$ , we plug-in  $x = 1$ , getting

$$y'(1) = 3 \cdot 1^2 + C = 3 + C$$

But  $y'(1) = 3$ , so we have the equation:

$$3 = 3 + C \quad .$$

Solving for  $C$ , we get  $C = 0$ . So now we know, for sure, that

$$y'(x) = 3x^2 \quad .$$

To get  $y(x)$ , we take the anti-derivative again:

$$y(x) = \int 3x^2 dx = 3\frac{x^3}{3} = x^3 + C \quad .$$

To find the exact value of (new)  $C$ , we plug-in  $x = 1$ , getting

$$y(1) = 1^3 + C = 1 + C$$

But  $y(1) = 1$ , so we have the equation:

$$1 = 1 + C \quad .$$

Solving for  $C$ , we get  $C = 0$ . So now we know for sure that

$$y(x) = x^3 + 0 = x^3 \quad .$$

**Ans. to 7b):**  $y = x^3$ .

**8.**

**Reminders:** For the definite integral

$$\int_a^b f(x) \quad ,$$

then the **Right-Approximation** with  $N$  intervals is (let  $h = \frac{b-a}{N}$ ), is:

$$R_N = h \sum_{j=1}^N f(a + jh)$$

and the **Left-Approximation** with  $N$  intervals is (let  $h = \frac{b-a}{N}$ ), is:

$$L_N = h \sum_{j=0}^{N-1} f(a + jh)$$

and the **Mid-Approximation** with  $N$  intervals is (let  $h = \frac{b-a}{N}$ ), is:

$$M_N = h \sum_{j=1}^N f(a + (j - \frac{1}{2})h)$$

(a) (4 points) Compute  $R_6$  for  $f(x) = x^2$  and the interval  $[0, 6]$ . Draw a picture of the region that  $R_6$  measures.

**Sol. to 8a):** Here  $N = 6$ , and  $h$  (a.k.a. as  $\Delta x$ ) is:  $(6 - 0)/6 = 1$ . Using the formula for  $R_N$ , given with  $N = 6$  and  $h = 1$ , we have

$$R_6 = 1 \cdot (f(1) + f(2) + f(3) + f(4) + f(5) + f(6)) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$$

**Ans. to 8a):**  $R_6 = 91$ .

(b) (4 points) Compute  $L_6$  for  $f(x) = x^2$  and the interval  $[0, 6]$ . Draw a picture of the region that  $L_6$  measures.

Using the formula for  $L_N$ , given with  $N = 6$  and  $h = 1$ , we have

$$L_6 = 1 \cdot (f(0) + f(1) + f(2) + f(3) + f(4) + f(5)) = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

**Ans. to 8b):**  $L_6 = 55$ .

(c) (4 points) Who is larger  $R_6$  or  $L_6$ ? Explain! (using the above pictures).

**Sol. to 8c):** From the picture, since  $y = x^2$  is **increasing**, the area of the region described by  $R_6$  **contains** the area under the curve  $y = x^2$ , whereas, the area of the region described by  $L_6$  **is contained** in the area under the curve  $y = x^2$ . So obviously  $R_6$  is larger than  $L_6$ .

**Note:** If the function would have been **decreasing**, then we would have the opposite/ If it is sometimes increasing, sometimes decreasing, then you can't tell beforehand.

### Extra Problems for Practice for the Final

Repeat the problem with

$$N = 4, f(x) = 2x + 2, [0, 4]$$

$$N = 4, f(x) = x^3, [0, 4]$$

$$N = 6, f(x) = 6 - x, [0, 6]$$