Solutions to the "QUIZ" for Oct. 27, 2008

1. Suppose that $f(x)$ is differentiable everywhere and we know that $f(2) = 1$ and $f'(x) \leq -2$ for all x.

a) What is the largest possible value for $f(4)$?

b) Show that $f(x)$ has a root in $[2, 4]$.

Solution to 1a): By the MVT there is a number c in the open interval $(2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2} \quad .$$

We know that $f(2) = 1$ so this means that

$$f'(c) = \frac{f(4) - 1}{2} \quad .$$

We are also told that $f'(x) \leq -2$ for *all* x, in particular $f'(c) \leq -2$, so we have

$$\frac{f(4) - 1}{2} \leq -2 \quad .$$

Multiplying both sides by 2:

$$f(4) - 1 \leq -4 \quad .$$

Adding 1 to both sides:

$$f(4) \leq -3 \quad .$$

Ans. to 1: The largest possible value for $f(4)$ is -3.

Solution to 1b) $f(x)$ is differentiable in $[2, 4]$ so it is automatically continous. At one end $x = 2$, we have $f(2) = 1$ that is **positive**, on the other end, at $x = 4$, $f(4) \leq -3$ so it is definitely **negative**. Since 0 is **intermediate** between any positive and any negative number, it follows by the **Intermediate Value Theorem** (IVT) that there is a number c in $(2, 4)$ such that $f(c) = 0$. This is our desired root.

Comments: Most people got a) right (except a few people messed up the algebra and the sign of the inequality). Only about half of the people got b) right. Please review IVT.

2. Find the critical points of $f(x) = 2x^3 - 9x^2 + 12x - 2$ and use the Second Derivative Test (if possible) to determine whether each corresponds to a local minimum or maximum.

Solution of 2. First we take the **first derivative**

$$f'(x) = 6x^2 - 18x + 12 \quad .$$

Now we set it equal to 0, and solve:

$$6x^2 - 18x + 12 = 0 \quad ,$$

factorizing:

$$6(x^2 - 3x + 2) = 0 \quad ,$$

$$6(x-1)(x-2) = 0 \quad ,$$

so the two **potential** candidates are $x = 1$ and $x = 2$.

Now we take the second derivative

$$f''(x) = 12x - 18 \quad .$$

and plug the candidates into it.

For $x = 1$, $f''(1) = 12 \cdot 1 - 18 = -6 < 0$, so

$x = 1$ is a local **maximum**.

For $x = 2$, $f''(1) = 12 \cdot 2 - 18 = 6 > 0$, so

$x = 2$ is a local **minimum**.

Ans. to 2: The critial points (numbers) are $x = 1$ (a local maximum) and $x = 2$ (a local minimum).

Comments: Most people got it right, but a quite a few people used the first derivative test.