Solutions to the "QUIZ" of Sept. 18, 2008

1. Explain why $x^4 + x - 3$ has a real root in the open interval $1 < x < 2$.

Solution:

We are going to use the Intermediate Value Theorem (IVT).

1) $f(x)$ is continuous (since it is a polynomial).

2) Plugging-in at the endpoints,

$$f(1) = 1^4 + 1 - 3 = -1 \quad ,$$

$$f(2) = 2^4 + 2 - 3 = 15 \quad ,$$

Since the output 0 lies between the outputs -1 (at $x = 1$) and the output 15 (at $x = 2$) it follows by IVT that somewhere in the open interval $(1, 2)$ there is a number c such that $f(c) = 0$. But being a **root** means exactly that, so IVT promises us a root (and of course it is real!).

Comments. About %70 of the people were on the right track (finding $f(1)$ and $f(2)$ and commenting that $f(x)$ is continuous), but only about %30 got it completely right. Quite a few people formulated the solution in a confused way, and sometimes it was pure gibberish.

2. Prove rigorously that

$$\lim_{x \to 1} 2x + 1 = 3 \quad .$$

Solution. Recall that the formal definition of

$$\lim_{x \to a} f(x) = L \quad ,$$

is that for every ϵ there is a δ such that

$$|f(x) - L| < \epsilon \quad if \quad |x - a| < \delta \quad .$$

We have to express δ as an expression of ϵ that will make it come true.

Let's implement the specifics. $f(x) = 2x + 1$, $L = 3$ and $x = a$.

$$|2x + 1 - 3| < \epsilon \quad if \quad |x - 1| < \delta \quad .$$

Using algebra,

$$|2x - 2| < \epsilon \quad if \quad |x - 1| < \delta \quad .$$

Using more algebra (factor 2 out)

$$|2(x - 1)| < \epsilon \quad if \quad |x - 1| < \delta \quad .$$

Using yet more algebra (take 2 out of the ||)

$$2|x-1| < \epsilon \quad if \quad |x-1| < \delta \quad .$$

Divide by 2

$$|x-1| < \epsilon/2 \quad if \quad |x-1| < \delta \quad .$$

Now both left sides correspond, so if we take

$$\delta = \frac{\epsilon}{2} \quad ,$$

things will work out.

Ans. We proved that for every $\epsilon > 0$ one can take $\delta = \epsilon/2$ such that

$$|f(x) - L| < \epsilon \quad if \quad |x-a| < \delta \quad .$$

This is the formal proof.

Comments Almost everyone got the expression $\delta = \epsilon/2$, and they would have gotten most of the credit, but to get full credit you had to phrase it in words.