Dr. Z's Math 152 Review Problems for Final Exam, Fall 2005

Due: Fri. Dec. 16, 4:00pm, 2005
(Bring to Final at Scott 123 along with notebook)

1. Let C be the curve $y = x^5/5$, with $0 \le x \le 3/4$. It is rotated about the x-axis.

(a) Set up an integral for the surface area of the solid formed.

(b) Using the binomial series and term-by-term integration, express the integral in part (a) as a convergent infinite series. Give numerical values for the first three terms in the series and a formula for the general term of the series.

(c) Explain why the method of (b) wouldn't work to find the length of the same curve extending from $x = 0$ all the way to $x = 3/2$. Give an approximate value for this length, using the Midpoint rule with $n = 3$ divisions.

(d) Given that $\left| \frac{d^2}{dx^2}(x^5\sqrt{1+x^8}) \right| \le 13$ for all $0 \le x \le 3/2$, estimate the error in your approximation in (c).

2. The curve with parametric equations

$$x = 17 - t^4, \quad y = 2 + 3\sin \pi t, \qquad -4 \le t \le 4$$

crosses itself at the point $(1, 2)$. Find the t values at which it crosses $(1, 2)$. Find the equations of both tangent lines at the point $(1, 2)$.

3. Find the solution of the differential equation $\frac{dy}{dx} = y\left(\frac{x^2 - 3x + 3}{x^2 - 3x + 2}\right)$ with $y(3) = 4$. Give an explicit formula for y as a function of x. Graph the solution and determine the largest interval $A < x < B$ for which the solution exists.

4. Let R be the region in the *fourth* quadrant which is bounded by the curves $y = -e^{-2x}$ and $y = 0$.

(a) Sketch the region R and find its area.

(b) Find the volume of the solids which result when the region R is revolved (1) about the x-axis; (2) about the y-axis. (Note that these integrals are improper.)

5. Calculate the following indefinite integrals:

(a) $\int e^x \sin x \, dx$ (b) $\int x^2\sqrt{9 - x^2}\, dx$ (c) $\int \sqrt{x^2 + 6x + 10}\, dx$

6. Find $\int_0^{\pi/2} \tan^3 x \, dx$ and $\int_0^{\pi/2} \sin^2 x \cos^5 x \, dx$.

7. Find $\int_0^{\pi/4} \sin^4 x \, dx$

8. Use geometric series to write the repeating decimals as a fraction. (a) $3.222222\ldots$ (b) $0.23444444444\ldots$ (c) $1.123123123123123\ldots$.

9. A certain radio active substance is known to have half-life 2000 years and to decay at a rate which is always proportional to the amount present. If a sample contains 10 grams of the substance today, how much will be left in 6000 years? How much was present in the sample 1000 years ago? Give exact answers, not decimal approximations.

10. A bacteria culture grows at a rate proportional to its size. If at the beginning there were 1000 bacteria, and after three hours there were 10000 bacteria, when will there be 1000000 bacteria?

11. (a) Does $\lim_{n\to\infty} \frac{\ln n}{n^3}$ exist? Explain your reasoning.

(b) Prove that $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges.

(c) Show that $\sum_{n=3}^{\infty} \frac{\ln n}{n^3} < \int_2^{\infty} \frac{\ln x}{x^3} \, dx$ by drawing areas related to the graph of $y = \frac{\ln x}{x^3}$.

12. Suppose you need numerical values of function $f(x)$ defined by a very complicated formula. You know, however, that $f(1) = 2$, $f'(1) = -3$, $f''(1) = 5$ and $f'''(1) = -1$. Moreover you know that the fourth derivative of $f(x)$ satisfies $|f''''(x)| \leq 10$ for all x in the interval $0 \leq x \leq 2$. Compute the third-degree Taylor polynomial T_3 for f centered at 1. Use it and Taylor's Inequality to solve the following problems.

(a) Calculate the best approximate value for $f(0.8)$ that you can from this information, and then estimate the error.

(b) Find a number $B > 0$ so that $|f(x) - T_3(x)| \leq 1/100$ for *all* numbers x in the interval $1 - B \leq x \leq 1 + B$.

13. Let $f(x) = \sqrt{1+x}$ and $g(x) = e^{2x}$.

(a) Find the coefficients a_0, a_1, a_2 in the Maclaurin series $f(x)g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$.

(b) Find the coefficients b_0, b_1, b_2 in the Maclaurin series $\frac{f(x)}{g(x)} = b_0 + b_1 x + b_2 x^2 + \cdots$.

(You may obtain your answers either by algebraic manipulation of known power series or by the definition of the Maclaurin series.)

14. Let $f(x) = e^{x^2}$ and $g(x) = \cos x$.

(a) Find the coefficients a_0, a_1, a_2, a_3 in the Maclaurin series $f(x)g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \cdots$.

(b) Find the coefficients b_0, b_1, b_2, b_3 in the Maclaurin series $\dfrac{f(x)}{g(x)} = b_0 + b_1x + b_2x^2 + b_3x^3 \cdots$.

(You may obtain your answers either by algebraic manipulation of known power series or by the definition of the Maclaurin series.)

15. Find the first three non-zero terms of the Maclaurin series for $f(x) = e^{\cos x}$.

16. Use the formula for the sum of a geometric series to calculate the Maclaurin series for the function

$$f(x) = \frac{x^2}{2 + 3x^4}.$$

Write your answer in sigma notation. Use the result to find an infinite series representation for $\displaystyle\int_0^1 f(t)\,dt$. Estimate the size of the difference between this integral and the 3rd partial sum of the series.

17. Determine if the following series are absolutely convergent, conditionally convergent, or divergent. In each case give details to support your answer and indicate which convergence test you are using.

(a) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n\ln n}$ (b) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{n^2}{n^3+4}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{5^n \cdot n^n}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{5^n \cdot n!}$

18. Use comparisons to determine whether the following improper integrals are convergent or divergent.

(a) $\displaystyle\int_0^{\infty}\frac{dx}{(x+3)(x+4)}$ (b) $\displaystyle\int_0^{\infty}\frac{dx}{(9+x^2)^{3/2}}$

19. Verify your answers to (a) and (b) in the preceding problem by calculating the integrals.

20. Find the exact value of

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

(Hint: use partial fractions)

21. Determine the radius and interval of convergence of each of the following power series. In addition, determine those points at which each series is absolutely convergent.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(n+1)}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(x+1)^n}{n2^n}$