

Dr. Z's Math152 Handout #10.4 [Areas and Length in Polar Coordinates]

By Doron Zeilberger

Problem Type 10.4a: Find the area of the region that is bounded by the given curve and lies in the specific sector

$$r = f(\theta) \quad , \quad \alpha \leq \theta \leq \beta,$$

where $f(\theta)$ is some expression in θ (representing a curve in polar coordinates), and α, β are some numbers (representing angles in radians)

Example Problem 10.4a: Find the area of the region that is bounded by the given curve and lies in the specific sector

$$r = \sin \theta \quad , \quad \pi/3 \leq \theta \leq 2\pi/3 \quad .$$

Steps

1. Set-up the integral for the area enclosed by a sector and a curve given in polar coordinates

$$Area = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$$

2. Evaluate the integral, using your Integration know-how.

Example

1.

$$Area = \frac{1}{2} \int_{\pi/3}^{2\pi/3} \sin^2 \theta d\theta$$

2.

$$\begin{aligned} Area &= \frac{1}{2} \int_{\pi/3}^{2\pi/3} \sin^2 \theta d\theta \\ &= \frac{1}{2} \int_{\pi/3}^{2\pi/3} \frac{1 - \cos(2\theta)}{2} d\theta \\ &= \frac{1}{4} \int_{\pi/3}^{2\pi/3} (1 - \cos(2\theta)) d\theta = \frac{1}{4} \left(\theta - \frac{\sin(2\theta)}{2} \right) \Big|_{\pi/3}^{2\pi/3} \\ &= \left(\frac{1}{4} \theta - \frac{1}{8} \sin(2\theta) \right) \Big|_{\pi/3}^{2\pi/3} \\ &= \frac{1}{4} (2\pi/3 - \pi/3) - \frac{1}{8} (\sin(4\pi/3) - \sin(2\pi/3)) \\ &= \pi/12 + \sin(\pi/3)/4 = \pi/12 + \sqrt{3}/8 \quad . \end{aligned}$$

Ans.: $\pi/12 + \sqrt{3}/8$.