Dr. Z's Math152 Handout #11.10 [Taylor and Maclaurin Series]

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Problem Type 11.10a: Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series.

Example Problem 11.10a: Find the Maclaurin series for $f(x) = \sin x$ using the definition of a Maclaurin series.

Steps	**Example**
1. Find the first few derivatives of $f(x)$. Then plug-in, $x = 0$.	**1.** $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x,$ $f'''(x) = -\cos x, f^{(4)}(x) = \sin x, f^{(5)}(x) = \cos x \ldots$ Plugging-in $x = 0$ we get $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1,$ $f^{(4)}(0) = 0, f^{(5)}(0) = 1, \ldots$
2. Write-down the general formula for the Maclaurin series and plug-in the values above. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.	**2.** $\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$ $\frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5 + \ldots$ $= \frac{0}{0!} + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \ldots$ $= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

3. If possible, detect a pattern and write the general series.

3.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Problem Type 11.10b: Find the Taylor series for $f(x)$ centered at the given value of a.

Example Problem 11.10b: Find the Taylor series for $f(x) = \sin x$ centered at $a = \pi/2$.

Steps | **Example**

1. Find the first few derivatives of $f(x)$. Then plug-in, $x = a$.

1.

$$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x,$$

$$f'''(x) = -\cos x, f^{(4)}(x) = \sin x, f^{(5)}(x) = \cos x \ldots$$

Plugging-in $x = \pi/2$ we get

$$f(\pi/2) = 1, f'(\pi/2) = 0, f''(\pi/2) = -1, f'''(\pi/2) = 0,$$

$$f^{(4)}(\pi/2) = 1, f^{(5)}(\pi/2) = 0, \ldots$$

2. Write-down the general formula for the Taylor series centered at $x = a$ and plug-in the values above.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad .$$

2.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n =$$

$$\frac{f^{(0)}(\pi/2)}{0!} + \frac{f^{(1)}(\pi/2)}{1!}(x-\pi/2) + \frac{f^{(2)}(\pi/2)}{2!}(x-\pi/2)^2 +$$

$$\frac{f^{(3)}(\pi/2)}{3!}(x-\pi/2)^3+\frac{f^{(4)}(\pi/2)}{4!}(x-\pi/2)^4+$$

$$\frac{f^{(5)}(\pi/2)}{5!}(x-\pi/2)^5+\dots$$

$$=\frac{1}{0!}+\frac{0}{1!}(x-\pi/2)+\frac{-1}{2!}(x-\pi/2)^2+\frac{0}{3!}(x-\pi/2)^3+$$

$$\frac{1}{4!}(x-\pi/2)^4+\frac{0}{5!}(x-\pi/2)^5+\dots$$

$$=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-\frac{(x-\pi/2)^6}{6!}+\dots$$

3. If possible, detect a pattern and write the general series.

3.

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{(x-\pi/2)^{2n}}{(2n)!}$$

Problem Type 11.10c: Use known Maclaurin series to obtain the Maclaurin series for $f(x)$, where $f(x)$ is a product and/or composition of standard functions.

Example Problem 11.10c: Use known Maclaurin series to obtain the Maclaurin series for $f(x)=x^3e^{-4x}$.

Steps

1. Decide who is (or are) the most important function in the expression, and write down its (their) Maclaurin series, using w rather than x.

Example

1. $f(x)=x^3e^{-4x}$ features the exponential function. Recall that

$$e^w=\sum_{n=0}^{\infty}\frac{w^n}{n!}\quad.$$

2. Find out what's inside the important function and plug-in for w the needed quantity.

2. Plugging-in $w = -4x$ into the exponential series, we get

$$e^{-4x} = \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} x^n$$

3. Use series manipulation to finish it up.

3.

$$f(x) = x^3 e^{-4x} = x^3 \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} x^{n+3} \quad .$$

Problem Type 11.10d: Use multiplication or division of power series to find the first four (or whatever) non-zero terms of the Maclaurin series for $f(x)$, where $f(x)$ is a product and/or quotinet of several standard functions.

Example Problem 11.10d: Find the first four non-zero terms of the Maclaurin expansion of $e^{2x}\cos(3x)$

Steps

1. Write the first few terms of the Maclaurin series of the 'ingredients' using the formula sheet or your memory.

Example

1.

$e^x = 1+x+x^2/2+x^3/6+\ldots \quad \cos x = 1-x^2/2+x^4/24+\ldots$

which yields

$e^{2x} = 1+2x+(2x)^2/2+(2x)^3/6+\ldots = 1+2x+2x^2+(4/3)x^3+\ldots \quad ,$

$\cos 3x = 1-(3x)^2/2+(3x)^4/24+\ldots = 1-(9/2)x^2+(27/8)x^4+\ldots$

2. Use algebra to multiply (or divide) the ingredients together, discarding all terms of higher order.

2.

$$e^{2x}\cos 3x$$

$= (1+2x+2x^2+(4/3)x^3+\ldots)(1-(9/2)x^2+(27/8)x^4+\ldots)$

$= (1-(9/2)x^2+(27/8)x^4+\ldots)+(2x)(1-(9/2)x^2+(27/8)x^4+\ldots)+$

$(2x^2)(1-(9/2)x^2+(27/8)x^4+\ldots)+(4/3)x^3(1-(9/2)x^2+(27/8)x^4+\ldots$

$= 1-(9/2)x^2+(27/8)x^4+\ldots+2x-9x^3+(27/4)x^3+\ldots$

$+2x^2-9x^4+\ldots+(4/3)x^3+\ldots$

3. Collect terms up to the desired power.

3.

$f(x) = 1+2x-(5/2)x^2+(97/12)x^3+\ldots$

Ans.: $f(x) = 1+2x-(5/2)x^2+(97/12)x^3+\ldots$.