Dr. Z's Math152 Handout #11.11 [The Binomial Series]

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Problem Type 11.11a: (a) Expand $\sqrt[m]{a+bx^n}$ (or $1/\sqrt[m]{a+bx^n}$) as a power series.

(b) Use part (a) to estimate some function-value correct to so-and-so many deciml places.

Example Problem 11.11a: (a) Expand $\sqrt[5]{1+x}$ as a power series.

(b) Use part (a) to estimate $\sqrt[5]{1.01}$ correct to six decimal places.

Steps	**Example**
1. First rewrite the function in pure exponent-notation $A(1+Bx^n)^k$ for some numbers A, B and k.	**1.** $f(x) = (1+x)^{1/5}$.
2. Write down the **Binomial Series**, either from your memory or from the formula sheet, using the variable w. Then replace w by whatever is needed to make it coincide with the $f(x)$. Spell out the first few terms. $$(1+w)^k = \sum_{n=0}^{\infty} \binom{k}{n} w^n \quad ,$$ where $$\binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} \quad .$$	**2.** $$\sqrt[5]{1+x} = (1+x)^{1/5} = \sum_{n=0}^{\infty} \binom{1/5}{n} x^n =$$ $$1+(1/5)x+\frac{(1/5)(-4/5)}{2!}x^2+\frac{(1/5)(-4/5)(-9/5)}{3!}x^3+ \frac{(1/5)(-4/5)(-9/5)(-14/5)}{4!}x^4+\dots$$ $$= 1+\frac{x}{5}x-\frac{2x^2}{25}+\frac{6x^3}{125}-\frac{21x^4}{625}+\dots$$ (this is the **Ans. to (a)**)

3. Decide which is the appropriate x to plug-in, and plug-it into the Maclaurin expansion, quit when the next-term-to-be-added (or subtracted) is less that the desired error.

3. Here $x = .01$, and we have the following approximations

$$\sqrt[5]{1.01} = 1+\frac{.01}{5}-\frac{2(.01)^2}{25}+\frac{6(.01)^3}{125}-\frac{21(.01)^4}{625}+\dots$$

$$1+(.2)10^{-2}-(.8)10^{-5}+(.48)10^{-7}+(.336)10^{-9}+\dots \quad .$$

Since our desired accuracy is 6 decimal places, it means that the allowed error is $(.5)10^{-6}$. The fourth term is already less than that, so we ignore it and anything after that, and we get

$$\sqrt[5]{1.01} \approx 1+(.2)10^{-2}-(.8)10^{-5} = 1.200200 \quad .$$

Ans. to (b): $\sqrt[5]{1.01} \approx 1.200200$

Problem Type 11.11b: (a) Use the binomial series to expand functions involving square-root, like $1/\sqrt{1-x^2}$, $1/\sqrt{1+x^2}$ etc.

(b) Use part (a) to find the Maclaurin series for some inverse-trig function that is known to be the indefinite integral of the function of part (a).

Example Problem 11.11b: (a) Use the binomial series to expand $1/\sqrt{1-x^2}$.

(b) Use part (a) to find the Maclaurin series for $\sin^{-1} x$.

Steps	**Example**
1. First rewrite the function in exponent notation $A(1+Bx^n)^k$ for some numbers A, B and k.	**1.** $f(x) = (1-x^2)^{-1/2}$.

2. Write down the **Binomial Series**, either from your memory or from the formula sheet, using w, and then replace w by the right monomial in x.

$$(1+w)^k = \sum_{n=0}^{\infty} \binom{k}{n} w^n \quad ,$$

where

$$\binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} \quad .$$

Use algebra to simplify $\binom{k}{n}$.

2. Replacing w by $-x^2$, and k by $-1/2$

$$(1-x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \binom{-1/2}{n} (-x^2)^n$$

We have

$$\binom{-1/2}{n} = \frac{(-1/2)(-3/2)\dots(-1/2-n+1)}{n!} =$$

$$\frac{(-1/2)(-3/2)\cdots(-(2n-1)/2)}{n!} =$$

$$\frac{(-1)^n(1)(3)\cdots(2n-1)}{2^n n!} =$$

And going back to the expansion of $(1-x^2)^{-1/2}$,

$$(1-x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!} x^{2n}$$

(this is the **Ans. to (a)**)

3. Integrate term-by-term, and plug-in $x=0$ to get C, and plug that C back.

3.

$$\sin^{-1} x = \int (1-x^2)^{-1/2} \, dx =$$

$$C + x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!} \int x^{2n} \, dx$$

$$C + x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!} \frac{x^{2n+1}}{2n+1}$$

$$= C + x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!(2n+1)} x^{2n+1}$$

When $x=0$, $\sin^{-1}(0)=0$ (since $\sin 0 = 0$), so $C=0$, and we have **Ans. to (b)**:

$$= x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!(2n+1)} x^{2n+1}$$