Dr. Z's Math152 Handout #11.3 [The Integral Test and Estimation of Sums]

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Problem Type 11.3a: Use the integral test to determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} f(n)$$

Example Problem 11.3a: Use the integral test to determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} ne^{-n}$$

Steps	**Example**
1. Find out whether the improper integral $$\int_1^{\infty} f(x)\ dx \quad ,$$ converges or diverges.	**1.** By integration by parts ($u = x$, $v = e^{-x}$) $$\int xe^{-x}\ dx \ = \ -e^{-x}(x+1) = -\frac{x+1}{e^x}$$ so $$\int_1^R xe^{-x} = -\frac{x+1}{e^x}\Big\vert_1^R$$ $$-\frac{R+1}{e^R} + \frac{2}{e} \quad .$$ Taking the limit as R goes to ∞ (by L'Hôpital): $$\lim_{R\to\infty}\left(-\frac{R+1}{e^R}+\frac{2}{e}\right) = -\lim_{R\to\infty}\frac{1}{e^R}+\frac{2}{e} == 0+\frac{2}{e} = \frac{2}{e} \quad .$$ Since this is finite, the improper integral **converges**.

2. If the corresponding improper integral converges, then the series converges. If the corresponding improper integral diverges, then the series diverges.

Warning: In case of convergence, the sum of the series is **not** the value of the integral! Usually there is no easy way to find the actual value of the sum (you can use a computer to find very good approximations), all that you can tell is whether the series converges or diverges.

2. Since the improper integral convegres, so does the seris.

Ans.: The series is convergent.

Problem Type 11.3b: Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{A}{n^p} \quad ,$$

where A and p are some numbers.

Example Problem 11.3b: Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{5}{n^{.97}} \quad .$$

Steps

1. A is irrelevant. By the **p-test** if $p > 1$ then the series converges, otherwise it diverges.

Tip: For future reference, you might wish to remember "Dr. Z's $p - q$ test":

$$\sum_{n=1}^{\infty} \frac{1}{n^p (\ln n)^q} \quad .$$

converges if $p > 1$ (regardless of q) and if $p = 1$ and $q > 1$. Otherwise not.

Example

1. Here $p = .97$ is ≤ 1, so the series is divergent.

Ans.: The series diverges by the p-test.

Problem Type 11.3c: Determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} a_n \quad ,$$

where a_n can be written, using algebra, as a combination of p-series.

Example Problem 11.3c: Determine whether the following series converges or diverges

$$(a) \quad \sum_{n=1}^{\infty} \frac{7+4\sqrt{n}}{n^3} \quad ,$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{7+4\sqrt{n}}{n^{4/3}} \quad .$$

Steps	**Example**
1. Use algebra to split them up,	**1.** For (a): $$(a) \quad \sum_{n=1}^{\infty} \frac{7+4\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{7}{n^3} + \sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^3}$$ $$7\sum_{n=1}^{\infty} \frac{1}{n^3} + 4\sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \quad .$$ For (b) $$(b) \quad \sum_{n=1}^{\infty} \frac{7+4\sqrt{n}}{n^{4/3}} = \sum_{n=1}^{\infty} \frac{7}{n^{4/3}} + \sum_{n=1}^{\infty} \frac{4(n^{1/2})}{n^{4/3}}$$ $$= 7\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} + 4\sum_{n=1}^{\infty} \frac{1}{n^{5/6}} \quad .$$

2. Decide for each, using the p-test, whether it is convergent or divergent. If they are all convergent, then the whole thing (the original series) is conevergent. If one (or more) of the components are divergent, then the original series is divergent.

2. For (a). We have two p-series, one with $p = 3$, the other with $p = 5/2$. Since these are both > 1, it follows by the p-test that they are both convergent. Hence the given series is cinvergent.

Ans. to (a): The series is convergent since it is a sum of two convergent p-series.

For (b). We have two p-series, one with $p = 4/3$, the other with $p = 5/6$. Since $5/6$ is ≤ 1, the second diverges by the p-test. Never mind that the first series is convergent, once you have one of them divergent, the whole thing is.

Ans. to (b): The series is divergent since it is a sum of two p-series, one of which is divergent.

Note: Of course if both p's are ≤ 1 then it has all the more reason to be divergent.

Problem Type 11.3d: Use an improper integral to find an integer N, so that the partial sum

$$S_N = \sum_{n=1}^{N} f(n)$$

is within ϵ of the sum of the whole infinite series $\sum_{n=1}^{\infty} f(n)$. Be sure to explain why the value of N you give is the correct answer. Do not evaluate S_N. In this kind of problems $f(x)$ has to be continuous, positive, and decreasing (i.e. must go down).

Example Problem 11.3d: Use an improper integral to find an integer N, so that the partial sum

$$S_N = \sum_{n=1}^{N} \frac{1}{n^3}$$

is within 10^{-7} of the sum of the whole infinite series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Be sure to explain why the value of N you give is the correct answer. Do not evaluate S_N.

Steps	**Example**

1. First make sure that the function is indeed going down, at least eventually (the easiest is by inspection of the curve, the safest is by taking the derivative $f'(x)$ and checking that it is always negative). In the problems that you are likely to get, this is almost always true (or it would be an unfair trick question). Then remember that the error $R_N = S - S_N$ satisfies

$$|R_n| \leq \int_N^\infty f(x)\ dx \quad .$$

1. $f(x) = 1/x^3$ is obviously 'nice' and decreasing for x positive (x^3 obviously goes up and up so its reciprocal goes down and down), so we are safe. W have

$$|R_N| \leq \int_N^\infty x^{-3}\ dx \quad .$$

2. Evaluate the improper integral, getting an expression in N.

2. $\int x^{-3}\ dx = x^{-2}/(-2)$ so

$$|R_N| \leq \int_N^\infty x^{-3}\ dx = \frac{-2}{x^2}|_N^\infty = \frac{-2}{(\infty)^2} - \frac{-2}{N^2} = \frac{2}{N^2} \quad .$$

3. Take that expression on the right, and set it $\leq \epsilon$, solve for N.

3. $2/N^2 \leq 10^{-7}$ means $N^2 \geq 2 \cdot 10^7$ which means $N \geq \sqrt{20} \cdot 10^3$ appx. 4472. So if we pick $N = 4473$ we are guaranteed to be within the prescribed error of 10^{-7}.