

Dr. Z's Math152 Handout #11.5 [Alternating Series]

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**Problem Type 11.5a:** Determine whether the series converges or diverges

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad ,$$

where  $b_n$  is always positive (or always negative).

**Example Problem 11.5a:** Determine whether the following series converge or diverge

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n^2 + 3} \quad , \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{1 + 3\sqrt{n}} \quad , \quad (c) \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \quad .$$

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**Steps**

1. First find  $\lim_{n \rightarrow \infty} b_n$ . If it does not exist, or even if it exists but is not 0, (in that case  $\lim_{n \rightarrow \infty} (-1)^n b_n$  wouldn't exist) then it follows **right away**, by the divergence test, that the series is divergent.

**Example**

1. For (a):

$$\lim_{n \rightarrow \infty} \frac{1}{5n^2 + 3} = 0 \quad ,$$

so we go on to step 2.

For (b):

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{1 + 3\sqrt{n}} = 1 \quad ,$$

since that limit is **not** 0, we are done!

**Ans. to (b):** The series is divergent by the divergence test.

For (c):

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0 \quad ,$$

so we proceed to step 2.

**2.** Try to show that  $b_n$  is decreasing, either by simple algebra, or if  $b_n$  is complicated and  $b_n = f(n)$  for some function  $f(x)$ , take the derivative  $f'(x)$  and show that it is negative (you would hardly ever have to do that).

**2.** For (a),  $n^2$  is obviously increasing, hence so is  $5n^2 + 3$  and hence its reciprocal  $\frac{1}{5n^2+3}$  is decreasing. We already know that its limit is 0. By the **alternating-series test** (nicknamed the **ping-pong test**), it follows that (a) is convergent.

**Ans. to (a):** Convergent by the ping-pong test.

For (c),  $\ln n$  is obviously increasing, hence its reciprocal,  $\frac{1}{\ln n}$  is decreasing. We already know that its limit is 0. By the **alternating-series test** (nicknamed the **ping-pong test**), it follows that (c) is convergent.

**Ans. to (c):** Convergent by the ping-pong test.