Dr. Z's Math152 Handout #6.3 [Volumes by Cylindrical Shells]

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Problem Type 6.3a: Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

$$y = f(x) \quad , \quad y = 0 \quad , \quad x = a, \quad x = b$$

Example Problem 6.3a: Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

$$y = 1/x^2 \quad , \quad y = 0 \quad , \quad x = 1 \quad , \quad x = 2$$

Steps

1. Sketc the region and make sure that indeed $x = a$ and $x = b$ (that are given by the problem) bound the region, and that the x-axis (alias $y = 0$) is below the curve $y = f(x)$. a and b are your *limits of integration*.

2. Set-up the integral with the folowing formula.

$$Volume = 2\pi \int_a^b x f(x) dx$$

3. Evaluate the integral.

Example

1. Sketching the region (do it!) shows that indeed $x = 1$ and $x = 2$ are the limits of integration.

2.

$$Volume = 2\pi \int_1^2 x \frac{1}{x^2} dx$$

3.

$$Volume = 2\pi \int_1^2 x \frac{1}{x^2} dx = 2\pi \int_1^2 \frac{1}{x} dx =$$

$$2\pi (\ln x)|_1^2 = 2\pi(\ln 2 - \ln 1) = 2\pi(\ln 2 - 0) = 2\pi \ln 2 \quad .$$

Ans.: $2\pi \ln 2$

Problem Type 6.3b: Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

$$y = f(x) \quad , \quad y = g(x) \quad .$$

Example Problem 6.3b: Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

$$y = 3 + 2x - x^2 \quad , \quad x + y = 3 \quad .$$

Steps

1. Sketch the two curves and find the region bounded by them. Unlike 6.3a, where the limits of integration a and b are given to you, here you must find them by solving for x the equation $f(x) = g(x)$. You should expect to get two roots. These are your *limits of integration* a and b. Look at the sketch and decide who is on TOP and who is at the BOTTOM.

2. Set-up the integral with the following formula.

$$Volume = 2\pi \int_a^b x(TOP - BOTTOM)dx$$

Example

1. In this case the second curve (that happens to be a line) is given *implictly*, and you must first convert it to *explicit* form $y = Expression(x)$. In this case, $x + y = 3$ becomes $y = 3 - x$. Solving $3 + 2x - x^2 = 3 - x$ yields $x^2 - 3x = 0$, which is $x(x-3) = 0$ giving the two roots $x = 0$ and $x = 3$. These are your *limits of integration*. From the sketch (do it!) $TOP = 3 + 2x - x^2$ and $BOTTOM = 3 - x$.

2.

$$Volume = 2\pi \int_0^3 x\left((3 + 2x - x^2) - (3 - x)\right) dx$$

3. Evaluate the integral.

3.

$$Volume = 2\pi \int_0^3 x\left((3 + 2x - x^2) - (3 - x)\right) dx$$

$$= 2\pi \int_0^3 x(3x - x^2)dx$$

$$= 2\pi \int_0^3 (3x^2 - x^3)dx = 2\pi(x^3 - x^4/4)|_0^3$$

$$= 2\pi((3^3 - 3^4/4) - 0) = 2\pi(27/4) = 27\pi/2 \quad .$$

Ans.: $\frac{27\pi}{2}$.