

Dr. Z's Math152 Handout #7.4

[Integration of Rational Functions by Partial Fractions]

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Problem Type 7.4a: Evaluate the integral

$$\int \frac{C_1x + C_2}{(x - a)(x - b)} dx \quad ,$$

where a, b, C_1, C_2 are numbers.

Example Problem 7.4a: Evaluate the integral

$$\int \frac{x - 1}{(x - 2)(x - 3)} dx \quad .$$

Steps

1. Express the integrand using *partial fractions*, by setting

$$\text{Integrand} = \frac{A}{x - a} + \frac{B}{x - b} \quad ,$$

where A and B are numbers to be determined shortly.

Example

1.

$$\frac{x - 1}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3} \quad .$$

2. Multiply both sides by the denominator of the integrand, and then plug-in **convenient values**, for x , and determine, in turn, A and B , and incorporate them in the expression for the integrand.

2.

$$x - 1 = A(x - 3) + B(x - 2) \quad ,$$

pluggin-in $x = 3$ gives,

$$3 - 1 = A \cdot 0 + B \cdot (3 - 2) = B \quad ,$$

so $B = 2$. Plugging-in $x = 2$ gives

$$2 - 1 = A \cdot (2 - 3) + B \cdot (2 - 2) = -A \quad ,$$

so $A = -1$, and we have

$$\frac{x - 1}{(x - 2)(x - 3)} = \frac{-1}{x - 2} + \frac{2}{x - 3} \quad .$$

3. Integrate, by doing each piece separately, using

$$\int \frac{dx}{x - c} = \ln|x - c|$$

with various c 's. Don't forget to add $+C$ at the very end.

3.

$$\begin{aligned} \int \frac{x - 1}{(x - 2)(x - 3)} dx &= \int \frac{-dx}{x - 2} + \int \frac{2 dx}{x - 3} \\ &= -\ln|x - 2| + 2\ln|x - 3| + C \end{aligned}$$

Ans.: $-\ln|x - 2| + 2\ln|x - 3| + C$.

Problem Type 7.4b: Evaluate the integral

$$\int \frac{C_1x^2 + C_2x + C_3}{(x - a)(x - b)^2} dx \quad ,$$

where a, b, C_1, C_2 are numbers.

Example Problem 7.4b: Evaluate the integral

$$\int \frac{x^2}{(x - 3)(x + 2)^2} dx \quad .$$

Steps

Example

1. Express the integrand using *partial fractions*, by setting

$$\text{Integrand} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2} ,$$

where A, B and C are numbers to be determined shortly.

2. Multiply both sides by the denominator of the integrand, and then plug-in **convenient values**, for x , and determine, in turn, A and C . To get B you equate the x^2 coefficient in both sides. Then incorporate these specific values for A, B, C that you obtained in the expression for the integrand.

1.

$$\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} .$$

2.

$$x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3) ,$$

plugging-in $x = 3$ gives,

$$3^2 = A(3+2)^2 + B \cdot (3-3)(3+2) + C \cdot (3-3)$$

so $9 = 25A$, and $A = 9/25$.

Plugging-in $x = -2$ gives

$$(-2)^2 = A(-2+2)^2 + B(-2-3)(-2+2) + C(-2-3) ,$$

so $4 = -5C$, and $C = -4/5$. We still need to find B . The coeff. of x^2 on the left is 1, and on the right is $A + B$, so $1 = A + B$, and so $B = 1 - A = 1 - 9/25 = 16/25$. Incorporating these values for A, B, C in Step 1 gives

$$\frac{x^2}{(x-3)(x+2)^2} = \frac{9/25}{x-3} + \frac{16/25}{x+2} + \frac{-4/5}{(x+2)^2} .$$

3. Integrate, by doing each piece separately, using

$$\int \frac{dx}{x-c} = \ln|x-c| \quad ,$$

$$\int \frac{dx}{(x-c)^2} = \frac{-1}{x-c} \quad .$$

with various c 's. Don't forget to add $+C$ at the very end.

3.

$$\int \frac{x^2}{(x-3)(x+2)^2} =$$

$$\int \frac{9/25}{x-3} dx + \int \frac{16/25}{x+2} dx + \int \frac{-4/5}{(x+2)^2} dx$$

$$(9/25) \ln|x-3| + (16/25) \ln|x+2| + \frac{4}{5(x+2)} + C$$

Ans.: $\frac{9}{25} \ln|x-3| + \frac{16}{25} \ln|x+2| + \frac{4}{5(x+2)} + C$.

Problem Type 7.4c: Evaluate the integral

$$\int \frac{C_1x^2 + C_2x + C_3}{(x-a)(x^2 + b^2)} dx \quad ,$$

where a, b, C_1, C_2, C_3 are numbers.

Example Problem 7.4c: Evaluate the integral

$$\int \frac{10}{(x-1)(x^2+9)} dx \quad .$$

Steps

1. Express the integrand using *partial fractions*, by setting

$$\text{Integrand} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b^2}$$

where A, B and C are numbers to be determined shortly.

Example

1.

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \quad .$$

2. Multiply both sides by the denominator of the integrand, and then plug-in the **convenient value** $x = a$, to get A . In this kind of problem there is only one (real) convenient value (if you know about complex numbers, then ib and $-ib$ are, but you are not supposed to know about these). So to get B and C you have to equate coefficients to get two equations.

2.

$$10 = A(x^2 + 9) + (Bx + C)(x - 1)$$

$$= (A + B)x^2 + (C - B)x + (9A - C) \quad ,$$

plugging-in $x = 1$ gives,

$$10 = A \cdot (1^2 + 9) + (B \cdot 1 + C) \cdot (1 - 1) = 10A$$

so $A = 1$.

Comparing the coeff. of x^2 and x on both sides gives $0 = A + B$, and $0 = C - B$ so $B = -A = -1$ and $C = B = -1$. Incorporating these values for A, B, C in Step 1 gives

$$\frac{10}{(x - 1)(x^2 + 9)} = \frac{1}{x - 1} + \frac{-x - 1}{x^2 + 9} \quad .$$

3. Integrate, by doing each piece separately, using

$$\int \frac{dx}{x - c} = \ln|x - c| \quad ,$$

$$\int \frac{2x dx}{x^2 + b^2} = \ln|x^2 + b^2| \quad ,$$

$$\int \frac{1}{x^2 + b^2} = \frac{1}{b} \tan^{-1} \frac{x}{b} \quad .$$

Dont forget to add $+C$ at the end.

3.

$$\int \frac{10}{(x - 1)(x^2 + 9)} dx = \int \frac{1}{x - 1} + \int \frac{-x - 1}{x^2 + 9} dx$$

$$\int \frac{1}{x - 1} dx - \int \frac{x}{x^2 + 9} dx - \int \frac{1}{x^2 + 9} dx$$

$$= \int \frac{dx}{x - 1} - \frac{1}{2} \int \frac{2x}{x^2 + 9} dx - \int \frac{dx}{x^2 + 9} =$$

$$\ln|x - 1| - \frac{1}{2} \ln|x^2 + 9| - \frac{1}{3} \tan^{-1} \frac{x}{3} + C \quad .$$

Ans.:

$$\ln|x - 1| - \frac{1}{2} \ln|x^2 + 9| - \frac{1}{3} \tan^{-1} \frac{x}{3} + C \quad .$$