Dr. Z's Math152 Handout #9.4 [Exponential Growth and Decay]

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Problem Type 9.4a:

A bacteria culture starts with B_0 bacteria and grows at a rate proportional to its size. After T_1 hours there are B_1 bacteria.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after T_2 hours.

(c) Find the rate of growth after T_2 hours.

(d) When will the population reach B_2?

Example Problem 9.4a:

A bacteria culture starts with 50 bacteria and grows at a rate proportional to its size. After 3 hours there are 800 bacteria.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 4 hours.

(c) Find the rate of growth after 4 hours.

(d) When will the population reach 3000?

Steps	**Example**
1. Let $P(t)$ be the number of bacteria after t hours. The initial value-diff. eq. is $P'(t) = kP(t) \quad , \quad P(0) = B_0 \quad ,$ where k is some yet-to-be-determined number. The solution that should be **memorized** is $P(t) = B_0 e^{kt} \quad .$	**1.** The initial value-diff. eq. is $P'(t) = kP(t) \quad , \quad P(0) = 50 \quad ,$ where k is some yet-to-be-determined number. The solution is $P(t) = 50e^{kt} \quad .$

2. Plug-in $t = T_1$ and use the fact that $P(T_1) = B_1$, and solve for k. Plug that k that you have just found into $P(t) = B_0 e^{kt}$, and simplify (if possible or convenient) using the ln rules $a \ln b = \ln(b^a)$ and $e^{\ln w} = w$.

Shortcut: In these problems, you can circumvent e and ln and use

$$P(t) = B_0 \left(\frac{B_1}{B_0}\right)^{t/T_1}$$

2. Plug-in $t = 3$ and use the fact that $P(3) = 800$, and solve for k.

$$P(3) = 800 \quad \textit{means} \quad 50e^{3k} = 800 \quad ,$$

so $e^{3k} = 800/50 = 16$. Taking ln we get $\ln(e^{3k}) = \ln(16)$ so $3k = \ln 16$ and $k = (1/3)(\ln 16) = \ln(16^{1/3})$.

Plugging into $P(t) = 50e^{kt}$. We get

$$P(t) = 50e^{(\ln(16^{1/3}))t} = 50(e^{\ln(16^{1/3})})^t = 50{\cdot}16^{t/3} \quad .$$

Ans. to (a):

$$P(t) = 50 \cdot 16^{t/3} \quad .$$

Using the shortcut we get faster

$$P(t) = B_0 \left(\frac{B_1}{B_0}\right)^{t/T_1} = 50 \left(\frac{800}{50}\right)^{t/3} = 50(16^{t/3}) \quad .$$

3. Plug-in $t = T_2$ into the formula for $P(t)$ that you have just found.

3.

$$P(4) = 50(16^{4/3}) \quad .$$

Ans. to (b): $50(16^{4/3})$ appx. 2016.

4. Since $P'(t) = kP(t)$, the rate of growth after T_2 hours is $kP(T_2)$, so multiply the answer to (b) by k from step 2 .

4. Rate of growth of bacteria after 4 hours, $P'(4)$, equals $kP(4) = (1/3)(\ln 16){\cdot}(50(16^{4/3}))$ appx. 1863 cell/hour.

5. Solve, for t, the equation $P(t) = B_2$	**5.** $50(16^{t/3}) = 3000 \quad \textit{means} \quad 16^{t/3} = 3000/50 = 60 \quad .$ Taking ln of both sides gives $\ln 16^{t/3} = \ln 60 \quad \textit{i.e.} \quad (t/3)\ln 16 = \ln 60 \quad ,$ giving $t = 3(\ln 60)/(\ln 16)$ which is appx. 4.43 hours, or 4 hours and 26 minutes. **Ans. to (d)**: $3(\ln 60)/(\ln 16)$ which is appx. 4 hours and 26 minutes.

Problem Type 9.4b: The half-life of some radioactive element is T_{half} years. Suppose that we have a M_0-mg sample.

(a) Find the mass that remains after t years.

(b) How much of the sample remains after T_1 years?

(c) After how long will only M_1 mg remain?

Example Problem 9.4b: The half-life of cesium-137 is 30 years. Suppose that we have a 100-mg sample.

(a) Find the mass that remains after t years.

(b) How much of the sample remains after 100 years?

(c) After how long will only 1 mg remain?

Steps	**Example**
1. The general formula for the mass after t years, let's call it $M(t)$ is $M(t) = M_0 \left(\frac{1}{2}\right)^{t/T_{half}}$	**1. Ans. to (a)**: $M(t) = 100 \left(\frac{1}{2}\right)^{t/30}$

2. Plug $t = T_1$ into the formula for $M(t)$ that you have just found.

2.

$$M(100) = 100\left(\frac{1}{2}\right)^{100/30} = 100\left(\frac{1}{2}\right)^{10/3} \quad appx. \quad 9.92mg$$

Ans. to (b): $100\left(\frac{1}{2}\right)^{10/3}$ *appx.* $9.92mg$.

3. Solve, for t, $M(t) = M_1$, using the expression for $M(t)$ that you found above.

3. We have to solve

$$100\left(\frac{1}{2}\right)^{t/30} = 1 \quad .$$

Dividing both sides by 100 we get

$$\left(\frac{1}{2}\right)^{t/30} = 1/100 \quad .$$

Taking reciprocals (optional)

$$2^{t/30} = 100 \quad .$$

Taking ln of both sides

$$(t/30)(\ln 2) = \ln 100 \quad ,$$

which gives $t = 30(\ln 100)/(\ln 2) = 60(\ln 10)/(\ln 2)$ appx. 199.3 years.

Ans. to (c): $60(\ln 10)/(\ln 2)$ appx. 199.3 years.

Problem Type 9.4c: When a cold drink is taken from a refrigerator, its temperature is T_0 degrees. After t_1 minutes in a $T_{ambient}$-degree room, its temperature has increased to T_1 degrees.

(a) What is the temperature of the drink after t_2 minutes?

(b) When will its temperature be T_2 degrees?

Example Problem 9.4c:

When a cold drink is taken from a refrigerator, its temperature is 5 degrees. After 25 minutes in a 20 degree room its temperature has increased to 10 degrees.

(a) What is the temperature of the drink after 50 minutes?

(b) When will its temperature be 15 degrees?

Steps	**Example**
1. Set up **Newton's Law of Cooling** $\frac{dT}{dt} = k(T - T_{ambient}) \quad , \quad T(0) = T_0$ where k is a constant, and $T_{ambient}$ is the ambient temperature. Taking $y = T - T_{ambient}$ this becomes $\frac{dy}{dt} = ky \quad , \quad y(0) = T_0 - T_{ambient}$ and remember that the solution is always $y(t) = y(0)e^{kt} \quad .$	**1.** Taking $y = T - 20$ (note for later that $T = y + 20$) we have $\frac{dy}{dt} = ky \quad , T(0) = 5 - 20 = -15 \quad ,$ whose solution is $y(t) = -15e^{kt} \quad .$
2. Find k by taking advantage of the fact that $y(t_1) = T_1 - T_{ambient}$. Plug $t = t_1$ into $y(t)$, set it equal to $T_1 - T_{ambient}$ and solve for k. Then plug that k into $y(t)$.	**2.** When t equals 25, T equals 10 so y equals $10 - 20 = -10$ and we have $y(25) = -10 \quad ,$ so $-15e^{25k} = -10 \quad i.e. \quad e^{25k} = 10/15 = 2/3 \quad .$ Taking the ln of both sides we have $25k = \ln(2/3) \quad \textit{that gives } k = (1/25)\ln(2/3) = -(1/25)\ln(3/2) \ .$ Plugging that k back into $y(t)$ we have a formula for $y(t)$: $y(t) = -15e^{(-t/25)\ln(3/2)} \quad .$

3. Plug $t = T_2$ into $y(t)$ to get y at that time, and add $T_{ambient}$ to get the actual temperature.

3.

$$y(50) = -15e^{(-50/25)\ln(3/2)} = -15e^{-2\ln(3/2)} = -15e^{\ln(3/2)^{-2}} =$$

$$= -15\cdot(3/2)^{-2} = -15\cdot(2/3)^2 - = -15\cdot(4/9) = -20/3 = -6\frac{2}{3} \quad .$$

To get the actual temperature we add 20, getting $T(50) = 20 - 6\frac{2}{3} = 13\frac{1}{3}$.

Ans. to (a): The temperature of the drink after 50 minutes was $13\frac{1}{3}{}^oC$.

4. Set $y(t)$ equal to $T_2 - T_{ambient}$ and solve for t.

4. Setting $y(t)$ equal to $15 - 20 = -5$ gives

$$-5 = -15e^{(-t/25)\ln(3/2)} \quad ,$$

which is the same as

$$\frac{1}{3} = e^{(-t/25)\ln(3/2)} \quad .$$

Taking reciprocals

$$3 = e^{(t/25)\ln(3/2)} \quad .$$

Taking ln:

$$\ln 3 = (t/25)\ln(3/2) \quad .$$

Solving for t we get

$$t = 25 \cdot \ln 3/\ln(3/2) \quad app. \quad 67.74 mins.$$

Ans. to (b): The temperature of the drink will be 15^oC after $25(\ln 3)/\ln(3/2)$ minutes which is approximately 67.74 minutes.