Dr. Z's Math152 Last Handout

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Problem Type Last1: A sequence is defined by the recurrence

$$a_1 = a \quad , \quad a_{n+1} = F(a_n) \quad for \quad n = 1, 2, 3, \ldots$$

Assume that this sequence converges. Find the possible values for $\lim_{n \to \infty} a_n$.

Example Problem Last1:

A sequence is defined by the recurrence

$$a_1 = 5, \quad a_{n+1} = 1 + 1/a_n \quad for \quad n = 1, 2, 3, \ldots$$

Assume that this sequence converges. Find the possible values for $\lim_{n \to \infty} a_n$.

Steps

1. We can write a computer program that compues the a_2, a_3, a_4 step-by-step and run it for n up to a million. But we are supposed to use calculus. Call $\lim_{n \to \infty} a_n$, x. and take the limits of both sides:

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} F(a_n)$$

Example

1. To compute Call $\lim_{n \to \infty} a_n$, x. and take the limits of both sides:

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} (1 + 1/a_n)$$

2. Since obviously $\lim_{n\to\infty} a_{n+1}$, is also x (when n goes to infinity, so does $n+1$) we have the **algebraic equation**

$$x = F(x) \quad ,$$

we solved for x and that is the answer(s).

2. Taking limits of both sides, we get

$$x = 1 + 1/x \quad ,$$

Simplifying, we get the **quadratic equation**

$$x^2 - x - 1 = 0$$

whose roots are

$$\frac{1 \pm \sqrt{5}}{2} \quad .$$

Ans.: The possible values of the limit are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. But we can rule out the second case, since a_n is obviously positive, and there is no way that its limit is negative. So **Final Ans.:** The limit is $(1+\sqrt{5})/2$.

Problem Type Last2: **Without** using L'Hôpital's Rule, compute

$$\lim_{x\to 0} COMPLICATED(x)$$

Example Problem Last2: **Without** using L'Hôpital's Rule, compute

$$\lim_{x\to 0} \frac{e^{3x} - 1 - 3x}{1 - \cos(2x)}$$

Steps | **Example**

1. You are supposed to use Maclaurin series. Find the first couple of terms and then take the limit.

1. Since $e^w = 1 + w + w^2/2 + w^3/6 + \ldots$,

$$e^{3x} = 1 + 3x + (9/2)x^2 + (9/16)x^3 + \ldots$$

so

$$e^{3x} - 1 - 3x =$$

$$1+3x+(9/2)x^2+(9/16)x^3-1-3x = (9/2)x^2+(9/16)x^3+\ldots$$

Regarding the denominator, since

$$cosw = 1 - w^2/2 + w^4/24 + \ldots \quad ,$$

we have

$$\cos(2x) = 1-(2x)^2/2+(2x)^4/24+\ldots = 1-2x^2+(2/3)x^4+\ldots$$

and

$$1-\cos(2x) = 1-(1-2x^2+(2/3)x^4+\ldots) = 2x^2-(2/3)x^4+\ldots =$$

So we have

$$\frac{e^{3x} - 1 - 3x}{1 - \cos(2x)} =$$

$$\frac{(9/2)x^2 + (9/16)x^3 + \ldots}{2x^2 - (2/3)x^4 + \ldots}$$

$$\frac{x^2[(9/2) + (9/16)x^{+} \ldots]}{x^2[2 - (2/3)x^2 + \ldots]}$$

$$= \frac{(9/2) + (9/16)x + \ldots}{2 - (2/3)x^2 + \ldots}$$

2. Now take the limit as $x \to 0$ by plugging, if possible $x = 0$. If the denominator (after simplifying) has a factor of x (or x^2, x^3 etc.), then the limit does not exist. If the denominator has no x factor, then, it is a simple plug-in of $x = 0$

2. Plugging-in $x = 0$ in

$$\frac{(9/2) + (9/16)x + \ldots}{2 - (2/3)x^2 + \ldots}$$

gives

$$\lim_{x \to 0} \frac{e^{3x} - 1 - 3x}{1 - \cos(2x)} = \frac{9/2}{2} = \frac{9}{4} \quad .$$

Ans.: The limit is 9/4.