MATH 152 (01-03), Dr. Z. , **Solutions to First Midterm**, Thurs. Oct. 20, 2005.

1. (10 points [5 each]) Find the following indefinite integrals
(a)

$$\int x^2 \sin x \, dx \quad .$$

Solution: Use Integration by parts with $u = x^2$, $v' = \sin x$. Then $u' = 2x$ and $v = -\cos x$.

$$\int x^2 \sin x \, dx = x^2(-\cos x) - \int 2x(-\cos x) \, dx = -x^2 \cos x + 2 \int x \cos x \, dx \quad .$$

We have to use integration by parts one more time and do the **subproblem** $\int x \cos x \, dx$.
Here we take $u = x$ and $v' = \cos x$, getting $u' = 1$ and $v = \sin x$ and so

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x \quad .$$

Going back to the main problem, we have

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx =$$

$$-x^2 \cos x + 2\,(x \sin x + \cos x) = (-x^2 + 2) \cos x + 2x \sin x + C \quad .$$

Ans. to 1(a) : $(-x^2 + 2) \cos x + 2x \sin x + C$.

(b)

$$\int \frac{3x}{x^2(x+1)} \, dx$$

Solution: First use algebra to simplify

$$\frac{3x}{x^2(x+1)} = \frac{3}{x(x+1)} \quad ,$$

so

$$\int \frac{3x}{x^2(x+1)} \, dx = \int \frac{3}{x(x+1)} \, dx \quad .$$

Now use **partial fractions**. Write

$$\frac{3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad .$$

Taking common denominators on the right we get

$$\frac{3}{x(x+1)} = \frac{A(x+1) + Bx}{x(x+1)} \quad .$$

Equating the numerators, we get

$$3 = A(x+1) + Bx \quad .$$

Plugging-in the convenient value $x = 0$, we get

$$3 = A \cdot (0+1) + B \cdot 0 = A \quad .$$

So $A = 3$. Plugging-in the convenient value $x = -1$, we get

$$3 = A \cdot (-1+1) + B \cdot (-1) = -B \quad .$$

So $B = -3$. Hence we get the partial-fraction decomposition

$$\frac{3}{x(x+1)} = \frac{3}{x} - \frac{3}{x+1} \quad .$$

Integrating, we get

$$\int \frac{3}{x(x+1)}\, dx = \int \left(\frac{3}{x} - \frac{3}{x+1} \right) dx =$$

$$3 \ln|x| - 3\ln|x+1| + C = 3 \ln |\frac{x}{x+1}| + C \quad .$$

Ans. to 1(b): $3\ln|x| - 3\ln|x+1| + C$ or $3\ln|\frac{x}{x+1}| + C$.

2. (10 points) The base of a solid is the region inside the ellipse $9x^2 + y^2 = 9$. Each cross section of the solid perpendicular to the y-axis is an equilateral triangle. What is the volume of the solid?
(Hint: the area of an equilateral triangle of side a is $\sqrt{3}a^2/4$.)

Solution: The cross section perpendicular to the y axis has length $a = 2\sqrt{1 - y^2/9}$ (by symmetry you have to double the x obtained by solving for x the equation $9x^2 + y^2 = 9$). The area of a cross-section $A(y)$ is

$$\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\left(2\sqrt{1 - \frac{y^2}{9}}\right)^2 = \sqrt{3}\left(1 - \frac{y^2}{9}\right) \quad .$$

We have to integrate from $y = -3$ to $y = 3$.

$$Volume = \int_{-3}^{3} \sqrt{3}(1 - \frac{y^2}{9})\, dy = \sqrt{3}\left(y - \frac{y^3}{27}\right)|_{-3}^{3} =$$

$$\sqrt{3}\left((3 - \frac{3^3}{27} - ((-3) - \frac{(-3)^3}{27})\right) = 4\sqrt{3} \quad .$$

Ans.: $4\sqrt{3}$.

3. (10 points, 5 each) Consider the curve $y = x^2/2$, $0 \le x \le 1$. Find (a) its length (b) the area of the surface formed by rotating it about the y-axis.
Solution: Recall that

$$Length = \int_a^b \sqrt{1 + f'(x)^2}\, dx \quad ,$$

$$S.A._y = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\, dx \quad .$$

Here $f(x) = x^2/2$. So $f'(x) = x$ and $\sqrt{1 + f'(x)^2} = \sqrt{1 + x^2}$ Hence

$$Length = \int_0^1 \sqrt{1 + x^2}\, dx \quad .$$

$$S.A._y = 2\pi \int_a^b x\sqrt{1 + x^2}\, dx \quad .$$

For the length we do the trig-substitution $x = \tan\theta$ that gives $dx = \sec^2\theta d\theta$ and $\sqrt{1 + x^2} = \sqrt{1 + \tan^2\theta} = \sqrt{sec^2\theta} = \sec\theta$. So

$$Length = \int_0^1 \sqrt{1 + x^2} = \int_0^{\pi/4} \sec^3\theta \quad .$$

For this we use integration by parts. $u = \sec\theta$, $v' = \sec^2\theta$, which gives $u' = \sec\theta\tan\theta$ and $v = \tan\theta$. So

$$Length = \int_0^{\pi/4} \sec^3\theta = \sec\theta\tan\theta|_0^{\pi/4} - \int_0^{\pi/4} \sec\theta\tan^2\theta\, d\theta =$$

$$\sec(\pi/4)\tan(\pi/4) - 0 - \int_0^{\pi/4} \sec\theta\tan^2\theta\, d\theta = \sqrt{2} - \int_0^{\pi/4} \sec\theta\tan^2\theta\, d\theta \quad .$$

Now comes a *trick*: using $\tan^2\theta = \sec^2\theta - 1$ we have

$$\int_0^{\pi/4} \sec^3\theta = \sqrt{2} - \int_0^{\pi/4} \sec\theta(\sec^2\theta - 1)\, d\theta = \sqrt{2} - \int_0^{\pi/4} \sec^3\theta + \int_0^{\pi/4} \sec\theta \quad .$$

Moving the $\int_0^{\pi/4} \sec^3\theta$ on the right to the left we have

$$2\int_0^{\pi/4} \sec^3\theta = \sqrt{2} + \int_0^{\pi/4} \sec\theta = \sqrt{2} + \ln(\sec\theta + \tan\theta)|_0^{\pi/4} =$$

$$\sqrt{2} + \ln(\sec\pi/4 + \tan\pi/4) - \ln(\sec 0 + \tan 0) = \sqrt{2} + \ln(\sqrt{2} + 1) - \ln(1) = \sqrt{2} + \ln(\sqrt{2} + 1)$$

We got

$$2\int_0^{\pi/4} \sec^3\theta = \sqrt{2} + \ln(\sqrt{2} + 1) \quad .$$

Finally, dividing by 2, we get:

$$\int_0^{\pi/4} \sec^3\theta = \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2} \quad .$$

Ans. to 2(a): $\frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2}$.
For the $S.A._y$ we have

$$S.A._y = 2\pi \int_0^1 x\sqrt{1+x^2}\,dx \quad .$$

Using the substitution $u = 1+x^2$ we get $du = 2xdx$

$$S.A. = 2\pi \int_0^1 x\sqrt{1+x^2}\,dx = 2\pi \int_1^2 \sqrt{u}\,du/2 = 2\pi(1/2)\int_1^2 \sqrt{u}\,du = \pi \int_1^2 u^{1/2}\,du =$$

$$\pi \frac{u^{3/2}}{3/2}|_1^2 = \frac{2\pi}{3}(2^{3/2} - 1^{3/2}) = \frac{2\pi}{3}(2\sqrt{2}-1)/3 \quad .$$

Ans. to (b): $\frac{2\pi(2\sqrt{2}-1)}{3}$.

4. (10 pts) I stretched a spring one meter from its natural length. Then you came along, and stretched it one meter further (from one meter to two meters [from the natural length]). Who did more work, and what is the ratio of the the work done by the harder worker to the work done by the easier worker? Explain!

Solution: The force is $F = kx$, where k is Hooke's constant.

$$MyWork = \int_0^1 kx\,dx = \frac{kx^2}{2}|_0^1 = \frac{k}{2} \quad .$$

$$YourWork = \int_1^2 kx\,dx = \frac{kx^2}{2}|_1^2 = \frac{k\cdot 2^2}{2} - \frac{k\cdot 1^2}{2} = \frac{3k}{2} \quad .$$

Ans. to (4): You did more work. The ratio of your work to my work is 3.

5. (10 points, 5 each) Determine whether each of the following integrals is convergent or divergent. Evaluate those that are convergent. Be sure to explain everything.
(a)

$$\int_e^{\infty} \frac{4\log x}{x^3}\,dx \quad .$$

Solution:

$$\int_e^{\infty} \frac{4\log x}{x^3} = \lim_{R\to\infty}\int_e^R \frac{4\log x}{x^3} \quad .$$

Let's use integration by parts with $u = \log x$ and $v' = 4x^{-3}$. Then $u' = 1/x$ and $v = -2x^{-2}$. So

$$\int_e^R \frac{4\log x}{x^3} = (\log x)(-2x^{-2})|_e^R - \int_1^R (-2x^{-2})(1/x)\,dx = -2\frac{\log R}{R^2} + 0 + 2\int_1^R x^{-3}\,dx =$$

$$-2(\log R)/R^2 + 0 + 2\frac{x^{-2}}{-2}|_e^R = -2\frac{\log R}{R^2} - \frac{1}{R^2} + \frac{1}{e^2}$$

Taking the limit we have

$$\int_e^\infty \frac{4\log x}{x^3} = -2\lim_{R\to\infty}\frac{\log R}{R^2} - \lim_{R\to\infty}\frac{1}{R^2} + \frac{1}{e^2}$$

The second limit is 0. For the first limit, we use L'Hôpital.

$$\lim_{R\to\infty}\frac{\log R}{R^2} = \lim_{R\to\infty}\frac{(\log R)'}{(R^2)'} = \lim_{R\to\infty}\frac{1/R}{2R} = \lim_{R\to\infty}\frac{1}{2R^2} = 0 \quad .$$

So the whole limit is $1/e^2$, a finite number.
Ans. to 5(a): The improper integral is convergent and its value is $1/e^2$.

(b)

$$\int_{10}^\infty \frac{x^{99} + x^{76} + 1}{x^{100} - x^{76} + 7}\,dx \quad .$$

Solution: By the limit comparison theorem, the convergence status of this integral is the same as that of

$$\int_{10}^\infty \frac{x^{99}}{x^{100}}\,dx = \int_{10}^\infty \frac{1}{x}\,dx \quad .$$

Now

$$\int_{10}^\infty \frac{1}{x}\,dx = \lim_{R\to\infty}\int_{10}^R \frac{1}{x}\,dx = \lim_{R\to\infty}(\ln x)|_{10}^R = \lim_{R\to\infty}[\ln R - \ln 10] = \infty - \ln 10 = \infty \quad .$$

Ans. to (b): The improper integral diverges, since the simplified intgeral diverges.

6. (10 pts) Find the average value of $\sin^2 x$ on the interval $0 \le x \le \pi/4$. Is it larger or smaller than the average of the maximum and minimum of $\sin^2 x$ on that interval? Draw a picture that explains this.

Solution:

$$Ave = \frac{1}{\pi/4 - 0}\int_0^{\pi/4} \sin^2 x\,dx = \frac{4}{\pi}\int_0^{\pi/4} \sin^2 x\,dx =$$

$$\frac{4}{\pi}\left(\int_0^{\pi/4}\frac{1-\cos 2x}{2}\,dx\right)=\frac{4}{\pi}\left(x/2-(\sin 2x)/4\right)|_0^{\pi/4}$$

$$=\frac{4}{\pi}[\pi/8-(\sin\pi/2)/4-0]=\frac{1}{2}-\frac{1}{\pi}\quad .$$

So the average of the function $f(x)=\sin^2 x$ is $\frac{1}{2}-\frac{1}{\pi}$. The two extremes are 0 and 1/2 and their average is 1/4. Since $\pi<4$ we have $1/\pi>1/4$ and so $1/2-1/\pi<1/4$. This means that the average of the function is less than that the average of its max and min. This, in turn, means that the curve of the function is *below* the line segment joining the beginning and end of $y=\sin^2 x$, $0\le x\le\pi/4$.

7. (10 pts [6 for (a) and 4 for (b)]) Let

$$I=\int_1^5\frac{1}{x}$$

(a) Use Simpson's rule with $n=4$ subdivisions to find an approximation, call it J.
Solution to 7(a): $\Delta x=(5-1)/4=1$, $x_0=1,x_1=2,x_2=3,x_3=4,x_4=5$.

$$S_4=\frac{1}{3}\left(1\cdot\frac{1}{1}+4\cdot\frac{1}{2}+2\cdot\frac{1}{3}+4\cdot\frac{1}{4}+1\cdot\frac{1}{5}\right)==\frac{73}{45}\quad .$$

Ans. to 7(a): $\frac{73}{45}$.
(b) Use the error estimate to find an upper bound for the error $|I-J|$.

$$|E_S|\le\frac{K(b-a)^5}{180n^4}=\frac{K(5-1)^5}{180\cdot 4^4}=\frac{K\cdot 4^5}{180\cdot 4^4}=\frac{K}{45}\quad ,$$

where K is the maximum of $f''''(x)$ in the interval $1\le x\le 5$. But $f(x)=1/x=x^{-1}$ so $f'(x)=-x^{-2}$ and $f''(x)=2x^{-3}$, $f'''(x)=-6x^{-4}$, and $f''''(x)=24x^{-5}=24/x^5$, whose maximum in the interval $[1,5]$ is $x=1$ and equals $24/1^5=24$. So $K=24$. We thus get

$$|E_S|\le\frac{K}{45}=\frac{24}{45}\quad .$$

Ans. to 7(b): The error is less than 24/45.

8. (10 points, 5 each) The region R is bounded by the curves $y=x^4$ and $y=\sqrt{x}$. Find the volume of the solid obtained by rotating R about: (a) about the x-axis (b) about the y-axis.
Solution: The common points are where $x^4=\sqrt{x}$. Squaring both sides: $x^8=x$, so $x(x^7-1)=0$ giving $x=0$ and $x=1$. Plugging in $x=1/2$ shows that $y=\sqrt{x}$ is on top (since $\frac{1}{\sqrt{2}}$ is larger than $\frac{1}{16}$).
For **a**,

$$Voluem=\pi\int_0^1\left((\sqrt{x})^2-(x^4)^2\right)\,dx=\pi\int_0^1\left(x-x^8\right)\,dx=$$

$$\pi\left(\frac{x^2}{2}-\frac{x^9}{9}\right)|_0^1=\pi\left(\frac{1}{2}-\frac{1}{9}\right)=\frac{7\pi}{18} \quad .$$

Ans. to 8(a): The volume is $7\pi/18$.

Regarding **8)b**,

$$Volume=2\pi\int_0^1 x(\sqrt{x}-x^4)\,dx=2\pi\int_0^1 x(x^{1/2}-x^4)\,dx=2\pi\int_0^1 (x^{3/2}-x^5)\,dx=$$

$$2\pi\left(\frac{x^{5/2}}{5/2}-\frac{x^6}{6}\right)|_0^1$$

$$=2\pi(\frac{2}{5}-\frac{1}{6})=\frac{7\pi}{15} \quad .$$

Ans. to 8(b): $\frac{7\pi}{15}$.

9. (10 pts) Solve the initial value problem

$$y'=\frac{y^2+1}{x} \quad ; \quad y(1)=1 \quad .$$

Solution:

$$\frac{dy}{dx}=\frac{y^2+1}{x}.$$

Cross-multiplying:

$$\frac{dy}{y^2+1}=\frac{dx}{x}$$

Integrating

$$\int\frac{dy}{y^2+1}=\int\frac{dx}{x}$$

giving,

$$\tan^{-1}y=\ln x+C$$

Plugging-in $x=1$, $y=1$:

$$\tan^{-1}1=\ln 1+C \quad .$$

which means

$$\pi/4=0+C \quad .$$

So $C=\pi/4$, and we have

$$\tan^{-1}y=\ln x+\pi/4 \quad .$$

Finally, solving for y by taking the tan of both sides

$$y=\tan(\ln x+\pi/4).$$

Ans to (9): $y = \tan(\ln x + \pi/4)$.

10. (10 pts) A bacteria culture starts with one thousand bacteria, and grows at a rate proportional to its size. After one hour there are ten thousand bacteria. (a) How many bacteria will there be after three hours? (b) When will there be 10^9 bacteria?

Solution: $P(t) = 1000 \cdot (10000/1000)^{t/1} = 1000 \cdot 10^t$.
After 3 hours we have $P(3) = 1000 \cdot 10^3 = 1000000$.
Ans. to 10(a): 1000000 bacteria.

For (b) solve $1000 \cdot 10^t = 10^9$, so $10^t = 10^6$. Equating the exponents of 10 on both sides, we get $t = 6$.
Ans. to 10(b): After 6 hours.