MATH 152 (07-09), Dr. Z. , **Solutions to First Midterm**, Thurs. Oct. 20, 2005.

1. (10 points [5 each]) Find the following indefinite integrals

(a)

$$\int e^x \sin x \, dx \quad ,$$

(b)

$$\int \frac{3x^2}{x(x^2+1)} \, dx \quad .$$

Solution of 1a): We integrate by parts with $u = e^x$ and $v' = \sin x$, getting $u' = e^x$ and $v = -\cos x$, yielding

$$\int e^x \sin x \, dx = (-\cos x)e^x - \int e^x(-\cos x)\, dx = -e^x \cos x + \int e^x \cos x \, dx \quad .$$

Now we have a **subproblem**: to find $\int e^x \cos x$. Using integration by parts one more time, this time with with $u = e^x$ and $v' = \cos x$, getting $u' = e^x$ and $v = \sin x$, yielding

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Plugging the answer to the subproblem in the main problem we have

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \quad .$$

Using **algebra** we move the integral on the right to the left, getting

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x \quad .$$

Finally, dividing both sides by 2 gives the final answer:

$$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2} = \frac{e^x(\sin x - \cos x)}{2} + C \quad .$$

Ans. to 1a): $\frac{1}{2}e^x(\sin x - \cos x) + C$.

Solution of 1b): Remember, **always try to use algebra before integrating!!!!**. Since

$$\frac{3x^2}{x(x^2+1)} = \frac{3x}{x^2+1} \quad ,$$

we have the **much simpler** integral

$$\int \frac{3x}{(x^2+1)} \, dx.$$

Now you ask yourself : "**is the top the derivative of the bottom?**" The derivate of the bottom is $2x$ which is 'close' so we first write it

$$\int \frac{2x}{(x^2+1)}\, dx$$

Then we ask ourselves: What do I have to multiply by to make it exactly the same? Obviously $3/2$, so we write

$$\int \frac{3x}{(x^2+1)} = \frac{3}{2}\int \frac{2x}{(x^2+1)}\, dx \quad .$$

Now this integral has the form where the top equals the derivative of the bottom and the answer is plainly $\ln|bottom|$, and we get

$$\frac{3}{2}\ln|x^2+1| + C \quad .$$

Ans. to 1(b): $\frac{3}{2}\ln|x^2+1| + C$.

Note: The 'official way' is to do a **substitution** $u = x^2 + 1$ in the (simplified!) integral. But please always remember: **alwyas try to use algebra first!**.

2. (10 points) The base of a solid is the region inside the curve $x^2 + y^{10} = 1$. Each cross section of the solid perpendicular to the y-axis is a rectangle whose length is twice its width, with the width at the bottom. What is the volume of the solid?

Solution to 2. The area of such a rectangle whose width is a and whose length is $2a$ is $2a^2$. Now we are doing cross-sections perpendicular to the **y**-axis, so we have to solve for x in terms of **y**!. We get $x = \sqrt{1-y^{10}}$, but by symmetry, a is twice that and $a = 2\sqrt{1-y^{10}}$ and the cross-section area $A(y)$ is

$$A(y) = 2a^2 = 2(2\sqrt{1-y^{10}})^2 = 8(1-y^{10}) \quad .$$

We integrate from $y = -1$ to $y = 1$ (to get these limits of integration we set $x = 0$ in $x^2 + y^{10} = 1$, getting $y^{10} = 1$ whose solutions are $y = -1$ and $y = 1$)

$$Volume = \int_{-1}^{1} A(y)\, dy = \int 8(1-y^{10})\, dy = 8\left(y - \frac{y^{11}}{11}\right)|_{-1}^{1} =$$

$$8\left(1 - \frac{1^{11}}{11}\right) - 8\left((-1) - \frac{(-1)^{11}}{11}\right) = 16\left(1 - \frac{1}{11}\right) = \frac{160}{11} \quad .$$

Ans. to 2): $\frac{160}{11}$.

3. (10 points, 4, 3, and 3 points resp.) Consider the curve $y = x^{10}$, $0 \le x \le 1$. Set-up, but do not evaluate integrals for (a) its length (b) the area of the surface formed by rotating it about the y-axis (c) the area of the surface formed by rotating it about the x-axis

Solution of (3): Remember:

$$Length = \int_a^b \sqrt{1+f'(x)^2}\,dx \quad ,$$

$$S.A._y = 2\pi \int_a^b x\sqrt{1+f'(x)^2}\,dx \quad ,$$

$$S.A._x = 2\pi \int_a^b y\sqrt{1+f'(x)^2}\,dx \quad .$$

It is always the **other guy** in the surface-area formulas. Note that the formula for $S.A_x$ may not be used as is, since it contains both y and x. We must replace the y by $f(x)$ getting the more explicit formula (that is in the formula sheet, note that the formula sheet does not have a formula for $S.A._y$)

$$S.A._x = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx \quad .$$

In our problem, $f(x) = x^{10}$, $a = 0$, and $b = 1$. $f'(x) = 10x^9$ and $\sqrt{1+f'(x)^2} = \sqrt{1+(10x^9)^2} = \sqrt{1+100x^{18}}$. So we have, **Ans. to 3a,b,c**, respectively:

$$Length = \int_0^1 \sqrt{1+100x^{18}}\,dx \quad ,$$

$$S.A._y = 2\pi \int_a^b x\sqrt{1+100x^{18}}\,dx \quad ,$$

$$S.A._x = 2\pi \int_a^b x^{10}\sqrt{1+100x^{18}}\,dx \quad .$$

4. (10 pts) A cable that weighs 2 kilograms per meter is used to lift a six hundered kilogram load up a mineshaft four hundred meters deep. Find the work done.

Solution to 4): The force is $F = (600+2x)g$, due to the fixed load (600) and the changing weight of the cable. When the cable is x meters deep, its weight is $2x$. It goes from $x = 400$ to $x = 0$, so the work is

$$Work = g\int_0^{400} (600+2x)\,dx \quad ,$$

Not computing this integral is a piece of cake.

$$Work = g\int_0^{400} (600+2x) = g(600x + x^2|_0^{400}) = g(600\cdot(400) + (400)^2) =$$

$$g(600+400)(400) = g(1000)(400) = 400000gJ \quad .$$

Ans. to 4: $400000g$ Joules. **Note:** you leave g as such, you do not replace it by 9.8 or 10).

5. (10 points, 5 each) Determine whether each of the following integrals is convergent or divergent. Evaluate those that are convergent. Be sure to explain everything.

(a)

$$\int_1^\infty \frac{x^2}{x^3+1}$$

(b)

$$\int_1^\infty xe^{-2x}$$

Solution of 5a). The straighforward way is to write

$$\int_1^\infty \frac{x^2}{x^3+1} = \lim_{R\to\infty} \int_1^R \frac{x^2}{x^3+1} == \lim_{R\to\infty} \frac{1}{3}\int_1^R \frac{3x^2}{x^3+1}$$

(using my favorite trick of getting it to be of the form 'top over bottom' with the top the derivative of the bottom). Continuing

$$= \lim_{R\to\infty} \frac{1}{3} \ln|x^3+1||_1^R = \lim_{R\to\infty} \frac{1}{3}\ln|R^3+1| - \frac{1}{3}\ln|1^3+1| = \infty \quad .$$

Since the limit is infinite, the improper integral **diverges**. **Ans. to 5a):** diverges.

Another way of doing it is with the limit comparison test. In the long run one may ignore 1 at the bottom and the convergence status of our integral is the same as

$$\int_1^\infty \frac{x^2}{x^3} = \int_1^\infty \frac{1}{x} \quad ,$$

which is far easier, and that diverges (since it equals $\ln\infty = \infty$). So we get the same answer: *diverges*.

Solution to 5b. Here we **must** use integration by parts and later on *L'Hôpital*. Integrating by parts

$$\int xe^{-2x}\,dx \quad ,$$

we pick $u = x$, $v' = e^{-2x}$. So $u' = 1$ and $v = e^{-2x}/(-2)$, and

$$\int xe^{-2x}\,dx = x(e^{-2x}/(-2)) - \int 1\cdot e^{-2x}/(-2)\,dx = -\frac{x}{2e^{2x}} - e^{-2x}/(4) =$$

$$-\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}} \quad .$$

Now

$$\int_1^\infty xe^{-2x}\,dx = \lim_{R\to\infty}\int_1^R xe^{-2x}\,dx = -\lim_{R\to\infty}\left(\frac{x}{2e^{2x}} + \frac{1}{4e^{2x}}\right)|_1^R$$

$$-\lim_{R\to\infty}\left((\frac{R}{2e^{2R}}+\frac{1}{4e^{2R}})-(\frac{1}{2e^2}+\frac{1}{4e^2})\right)=\frac{3}{4e^2}-\lim_{R\to\infty}\left(\frac{R}{2e^{2R}}+\lim_{R\to\infty}\frac{1}{4e^{2R}}\right) \quad .$$

The second limit is obviously $1/e^{\infty}=1/\infty=0$, while for the second, we need L'Hôpital:

$$\lim_{R\to\infty}\frac{R}{2e^{2R}}=\lim_{R}\to\infty\frac{1}{4e^{2R}}=\frac{1}{4e^{2\infty}}=\frac{1}{\infty}=0 \quad .$$

So the answer to the whole thing is $\frac{3}{4e^2}$, which is a finite number, hence the integral **converges**.

Ans. to 5b): The improper integral converges and its value is $\frac{3}{4e^2}$.

6. (10 pts) Find the average value, f_{ave}, of $f(x)=x^3$ on the interval $0\le x\le 1$. Find a number c such that $f(c)=f_{ave}$.

Solution to 6, first part:

$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)\,dx \quad .$$

Here $f(x)=x^3$ and $a=0$, $b=1$ and we have

$$f_{ave}=\frac{1}{1-0}\int_0^1 x^3=\frac{x^4}{4}|_0^1=\frac{1^4}{4}-\frac{0^4}{4}=\frac{1}{4} \quad .$$

Ans. to first part of 6: 1/4.

For the second part, we solve, for c, the equation $f(c)=f_{ave}$. So we have to solve

$$c^3=\frac{1}{4} \quad ,$$

whose solution is

$$c=\frac{1}{4^{1/3}} \quad ,$$

7. (10 pts [6 for (a) and 4 for (b)]) Let

$$I=\int_1^2 4x^3$$

(a) Use the Trapezoid rule with $n=4$ subdivisions to find an approximation, call it J.

(b) Use the error estimate to find an upper bound for the error $|I-J|$.

Solution to 7(a): Here $n=4$ and $\Delta x=(2-1)/4=1/4$, $x_0=1$, $x_1=5/4$, $x_2=6/4=3/2$, $x_3=7/4$, $x_4=2$ so

$$T_4=\frac{1/4}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right)=$$

$$\frac{1}{8}\left(f(1)+2f(5/4)+2f(3/2)+2f(7/4)+f(2)\right)=$$

$$\frac{1}{8}\left(4\cdot 1^3+2\cdot 4\cdot(5/4)^3+2\cdot 4\cdot(3/2)^3+2\cdot 4\cdot(7/4)^3+1\cdot 4\cdot(2)^3\right)$$

$$\frac{4}{8}\left(1^3+2\cdot(5/4)^3+2\cdot(3/2)^3+\cdot(7/4)^3+1\cdot(2)^3\right)=\frac{243}{16} \quad .$$

Ans. to 7a): $\frac{243}{16}$.

Solution to 7b):

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad ,$$

where K is the maximum of $|f''(x)|$ in the interval $[a,b]$. Here $n=4$ and $a=0$ and $b=1$, so

$$|E_T| \le \frac{K(1-0)^3}{12\cdot 4^2}=\frac{K}{192} \quad .$$

It remains to find K. $f(x)=4x^3$ so $f'(x)=12x^2$ and $f''(x)=24x$. The maximum of $f''(x)$ on the interval $[1,2]$ is at the endpoint $x=2$ (there are no local max or min since $(24x)'=24$ is never zero). So $K=24\cdot 2$. Combining, we get

$$|E_T| \le \frac{K(1-0)^3}{12\cdot 4^2}=\frac{48}{192}=\frac{1}{4} \quad .$$

Ans. to 7b): The error $|I-J|$ is less than $\frac{1}{4}$.

8. (10 points, 5 each) The region R is bounded by the curves $y=\sin x$ and $y=\cos x$ and the y axis for $0 \le x \le \pi/4$. Find the volume of the solid obtained by rotating R about the x-axis.

solution to 8: The volume is

$$\pi\int_a^b (TOP^2-BOT^2)dx$$

A rough sketch, or plugging-in, shows that $y=\cos x$ is on top of $y=\sin x$. The limits are kindly given by the problem $a=0$ and $b=\pi/4$.

$$Volume=\pi\int_0^{\pi/4}(\cos^2 x-\sin^2 x)dx$$

Now, many people split it into two parts $\int\cos^2 x$ and $\int\sin^2 x$, and some did it correctly, but it took them a long time, and some messed up the algebra. But you should have **pattern-recognition**

$$\cos^2 x-\sin^2 x$$

should **ring a bell**, it is a **trig identity**,

$$\cos^2 x-\sin^2 x=\cos 2x \quad ,$$

so now the rest is easy

$$Volume = \pi \int_0^{\pi/4} \cos(2x)dx = \pi \frac{\sin(2x)}{2}|_0^{\pi/4} = \pi \frac{\sin(2\pi/4)}{2} - \pi \frac{\sin(2 \cdot 0)}{2} = \pi \cdot \frac{\sin(\pi/2)}{2} = \pi \frac{1}{2} = \frac{\pi}{2} \quad .$$

Ans. to 8): $\pi/2$.

9. (10 pts) Find the values of r for which $y = e^{rt}$ satisfies the differential equation

$$y''' - 5y'' + 6y' = 0 \quad .$$

Solution to 9: $y = e^{rt}$. Remember r is **number**! So $y' = re^{rt}$, $y'' = r^2e^{rt}$, $y''' = r^3e^{rt}$, and plugging these into the diff. eq.

$$y''' - 5y'' + 6y' = r^3e^{rt} - 5r^2e^{rt} + 6re^{rt} = e^{rt}(r^3 - 5r^2 + 6r) = 0$$

Now the exponential function is never zero, so it is OK to divide by it getting the **algebraic equation**, in r,

$$r^3 - 5r^2 + 6r = 0 \quad .$$

Factoring (correctly!, it is amazing how many people mess up this Algebra I task)

$$r(r-2)(r-3) = 0 \quad ,$$

whose solutions are $r = 0, r = 2, r = 3$.

Ans. to 9): $r = 0, r = 2, r = 3$.

10. (10 pts) The half-life of Zeilbergerium is 1000 years. If right now you have a sample of 160 pounds, how long would it take for there to be only 10 pounds left? How much would be left after 10000 years?

Solution to 10: Recall that if M_0 is the initial mass, and T_h is the half-life, then

$$M(t) = M_0(\frac{1}{2})^{t/T_h} \quad ,$$

so

$$M(t) = 160 \cdot \left(\frac{1}{2}\right)^{t/1000} \quad ,$$

To find out when there would be 10 pounds we solve

$$10 = 160 \cdot \left(\frac{1}{2}\right)^{t/1000} \quad ,$$

and do the algebra

$$\frac{10}{160} = \left(\frac{1}{2}\right)^{t/1000} \quad ,$$

so

$$\frac{1}{16} = \left(\frac{1}{2}\right)^{t/1000} \quad ,$$

Now you could take ln of both sides, but it is easier to write $16 = 2^4$ and get

$$(\frac{1}{2})^4 = \left(\frac{1}{2}\right)^{t/1000} \quad ,$$

Equating the exponent of $\frac{1}{2}$ we get

$$4 = t/1000 \quad ,$$

yielding $t = 4000$.

Ans. to first part of 10: 4000 years.

The second part is even easier. Plugging-in $t = 10000$ into $M(t)$ above, we get

$$M(10000) = 160 \cdot \left(\frac{1}{2}\right)^{10000/1000} = \frac{160}{1024} = \frac{5}{32} \quad .$$

Ans. to second part of 10: $\frac{5}{32}$.