MATH 152 (01-03), Dr. Z. , **Solution to Second Midterm**, Monday Nov. 28, 2005.

1. (10 points [5 each]) For each of the two series below, determine whether they converge or diverge .

$$(a)\ \sum_{n=1}^{\infty}(-2)^n \quad , \quad (b)\ \sum_{n=1}^{\infty}\frac{n^5}{5n^5+6n^4+8} \quad ,$$

Solution of 1(a): $\lim_{n\to\infty} 2^n = \infty$ so $\lim_{n\to\infty}(-2)^n$ definitely does not exist, so **divergent by divergence test**.

Another way: This is a geometric series with $r=-2$, since $|-2| \geq 1$ it is **divergent by geometric series test**.

Yet Another way: By the ratio test, the limit of the ratio is -2, so $\rho = 2$ and since since $2 \geq 1$ it is **divergent by ratio test**.

Yet Another way: By the root test. Since $|a_n|^{1/n} = (2^n)^{1/n} = 2$. So $\lim_{n\to\infty}|a_n|^{1/n} = 2$ so $\rho = 2$ and since since $2 \geq 1$ it is **divergent by root test**.

Solution of 1(b): The limit of a_n is $1/5$, which is not zero, so **divergent by divergence test**.

Another solution: By the Limit Comarison test ("forget about the little ones"), this has the same convergence-status as

$$\sum_{n=1}^{\infty}\frac{n^5}{5n^5} = \sum_{n=1}^{\infty}\frac{1}{5} = \infty \quad ,$$

since the simplified version version diverges (since it equals ∞) the original one does too (by Limit Comparison)

2. (10 points, 5 each) Determine whether the following series converge or diverge. Explain what test(s) you are using.

$$(a)\quad \sum_{n=1}^{\infty}\frac{7+14\sqrt{n}}{n^2} \quad ,$$

$$(b)\quad \sum_{n=1}^{\infty}\frac{7+4n}{n^{5/3}} \quad .$$

Solution to 2(a) By the Limit Comparison Test,

$$\sum_{n=1}^{\infty} \frac{7 + 14\sqrt{n}}{n^2} \quad ,$$

has the same convergence-status as

$$\sum_{n=1}^{\infty} \frac{14\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{14n^{1/2}}{n^2} = \sum_{n=1}^{\infty} \frac{14}{n^{3/2}} = 14 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad ,$$

which converges by the p-test ($p = 3/2$). Hence: **converges by Limit-Comparison and p-test**.

Solution to 2(b) By the Limit Comparison Test,

$$\sum_{n=1}^{\infty} \frac{7 + 4n}{n^{5/3}} \quad .$$

has the same convergence-status as

$$\sum_{n=1}^{\infty} \frac{4n}{n^{5/3}} = 4 \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

which diverges by the p-test ($p = 2/3$). Hence: **diverges by Limit-Comparison and p-test**.

3. Use an improper integral to find an integer N, so that the partial sum

$$S_N = \sum_{n=1}^{N} \frac{1}{n^4}$$

is within 10^{-10} of the sum of the whole infinite series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Be sure to explain why the value of N you give is the correct answer. Do not evaluate S_N.

Solution to (3):

$$|Error| \leq \int_{N}^{\infty} \frac{1}{x^4} \, dx$$

Common Error: NOT $\int_1^N \frac{1}{x^4} \, dx$

$$|Error| \le \int_N^\infty \frac{1}{x^4}\,dx = \int_N^\infty x^{-4} = \frac{x^{-3}}{-3}|_N^\infty = -\frac{1}{3x^3}|_N^\infty =$$

$$-\frac{1}{3\cdot\infty^3} - -\frac{1}{3\cdot N^3} = 0 + \frac{1}{3\cdot N^3} = \frac{1}{3\cdot N^3}$$

We need

$$\frac{1}{3\cdot N^3} \le 10^{-10}$$

Taking **reciprocals** (**Warning: Don't forget to reverse the inequality!**),

$$3N^3 \ge 10^{10}$$

and solving for N

$$N^3 \ge \frac{10^{10}}{3}$$

$$N \ge (\frac{10^{10}}{3})^{1/3} = (\frac{10}{3})^{1/3} \cdot 1000$$

Ans. to (3): $N \ge (\frac{10}{3})^{1/3}1000$, so the least possible N is $\frac{10}{3})^{1/3}1000$.

4. (10 points, 3,3,4, resp.) Determine whether the following series converge or diverge (a) $\sum_{n=1}^\infty \frac{9+11n}{(n^2+1)^3}$, (b) $\sum_{n=1}^\infty \frac{3+2^n}{5+5^n}$, (c) $\sum_{n=1}^\infty \frac{n^4+6}{\sqrt{n^{10}+11}}$.

Solution to 4(a) : By the Limit Comparison Test ("forget about the little ones") this has the same convergence status as the simplified series

$$\sum_{n=1}^\infty \frac{11n}{(n^2)^3} = \sum_{n=1}^\infty \frac{11n}{n^6} = 11\sum_{n=1}^\infty \frac{1}{n^5} \quad ,$$

and this converges by the p-test ($p = 5$ bigger than 1).

Hence: **Ans.**: Converges by Limit Comparison Test and p-test.

Solution to 4(b) : By the Limit Comparison Test ("forget about the little ones") this has the same convergence status as the simplified series

$$\sum_{n=1}^\infty \frac{2^n}{5^n}$$

(in the long-run, 3 is nothing next to 2^n and 5 is nothing next to 5^n).

Now this simplified series is

$$\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n \quad ,$$

which is a geometric series with $r = 2/5$, hence converges by the geometric series test (since $|r| < 1$).

Hence **Ans.**: converges by Limit Comparison Test and geometric series test ($r = 2/5$).

Solution to 4(c) : By the Limit Comparison Test, this has the same convergence status as the simplified series:

$$\sum_{n=1}^{\infty} \frac{n^4}{\sqrt{n^{10}}} = \sum_{n=1}^{\infty} \frac{n^4}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n} \quad ,$$

which diverges by the p-test ($p = 1$). **Ans.**: Diverges by Limit Comparison Test and p-test ($p = 1$).

5. (10 points, 3,3,4, resp.) Determine whether the following series converge absolutely, converge conditionally or diverge

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1} \quad , \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{1+2\sqrt{n}} \quad , \quad (c) \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^5} \quad .$$

Solution to 5(a): FIRST, consider the **absolute version**

$$\sum_{n=1}^{\infty} \frac{(1)^n}{n^3+1} = \sum_{n=1}^{\infty} \frac{1}{n^3+1}$$

By the Limit Comparison Test this has the same convergence status as the simplified series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad ,$$

which converges by the p-test ($p = 3$). Hence the abslute version converges, and hence **Ans.**: Series converges abslutely.

Warning: A common mistake is to try and use the Alternating Series Test. You never use it unless you have to. If it converges absolutely, then the problem is over and we never have to worry about using it.

Solution to 5(b): By the Limit Comparison Test this has the same convergence status as

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{2\sqrt{n}} = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n$$

and the latter diverges by the **divergence test** (since $(-1)^n$ does not have a limit).

Solution to 5(c): FIRST, consider the **absolute version**

$$\sum_{n=2}^{\infty} \frac{(1)^n}{(\ln n)^5} \quad = \sum_{n=2}^{\infty} \frac{1}{(\ln n)^5} \quad ,$$

and this diverges by Dr. Z's $p-q$ test ($p=0$ $q=5$) (note that p is not 5, $p=0$ since there is no mention of n so it is like n^0) So we know that it is **not absolutely convergent**.

Now we have to try to use the Alternating Series Test. $1/(\ln n)^5$ is decreasing (since $\ln x$ is an increasing function so its reciprocal is decreasing and so is its fifth power). Also it goes to zero as $n \to \infty$ (since $\ln x$ goes to ∞). It follows by the Alternating Series Test that the original series is convergent. Since it is not absolutely convegent, it must be conditionally convergent.

Ans.: Series converges conditionally

6. (10 points, 5 each) Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

$$(a) \sum_{n=1}^{\infty} \frac{(-2)^n n!}{n^n} \quad .$$

$$(b) \sum_{n=1}^{\infty} \frac{n^n}{3^n n!} \quad .$$

Solution of 6(a). Whenever $n!$ shows up, we must use the **ratio test**.

It is always safer to take the **absolute values**.

$$|a_n| = \frac{2^n n!}{n^n}$$

So

$$|a_{n+1}| = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}$$

And the ratio is

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = \frac{2^{n+1}(n+1)! n^n}{2^n n! (n+1)^{n+1}}$$

$$\frac{2^n 2^1 (n+1)! n^n}{2^n n! (n+1)^{n+1}} = \frac{2(n+1)n^n}{(n+1)^{n+1}} = \frac{2n^n}{(n+1)^n}$$

Now we need to take limits. **Warning:** You can not use “forget about the little ones” since it is inside a power of n.

You need the famous limit

$$\lim_{n\to\infty} (\frac{n+1}{n})^n = e \quad ,$$

or rather its reciprocal

$$\lim_{n\to\infty} (\frac{n}{n+1})^n = \frac{1}{e} \quad ,$$

So

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2n^n}{(n+1)^n} = 2 \lim_{n\to\infty} (\frac{n}{n+1})^n = \frac{2}{e} \quad .$$

But $e = 2.71828...$ so $\rho < 1$ and we get **Ans.**: Converges absolutely by ratio test.

Solution of 6(b). Whenever $n!$ shows up, we must use the **ratio test**.

It is always safer to take the **absolute values**.

$$|a_n| = \frac{n^n}{3^n n!}$$

So

$$|a_{n+1}| = \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}$$

And the ratio is

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{n^n}{3^n n!}} = \frac{(n+1)^{n+1}3^n n!}{3^{n+1}(n+1)!n^n} = \frac{(n+1)^{n+1}}{3(n+1)n^n} = \frac{(n+1)^n}{3n^n} = \frac{1}{3}(\frac{n+1}{n})^n$$

Now for taking the limit, you need the famous limit

$$\lim_{n\to\infty} (\frac{n+1}{n})^n = e \quad ,$$

So

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{3} \lim_{n\to\infty} (\frac{(n+1)}{n})^n = \frac{e}{3} \quad .$$

But $e = 2.71828...$ so $\rho < 1$ and we get **Ans.**: Converges absolutely by ratio test.

7. (10 points) Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{n^3 5^n} \quad .$$

Solution of 7: You can use either the ratio test or root test. Let's use the ratio test.

$$a_n = \frac{(x-4)^n}{n^3 5^n} \quad ,$$

Replacing n by $n+1$ gives

$$a_{n+1} = \frac{(x-4)^{n+1}}{(n+1)^3 5^{n+1}} \quad ,$$

Taking the ratio a_{n+1}/a_n and simplifying,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(x-4)^{n+1}}{(n+1)^3 5^{n+1}}}{\frac{(x-4)^n}{n^3 5^n}} = \frac{(x-4)^{n+1} n^3 5^n}{(n+1)^3 5^{n+1} (x-4)^n}$$

$$= \frac{(x-4)^n (x-4) n^3 5^n}{(n+1)^3 5^n \cdot 5 (x-4)^n} = \frac{(x-4) n^3}{(n+1)^3 \cdot 5} = \frac{(x-4)}{5} \cdot \frac{n^3}{(n+1)^3} \quad .$$

This is the **simplified ratio**. Taking the limit as $n \to \infty$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(x-4)}{5} \cdot \frac{n^3}{(n+1)^3} =$$

$$\frac{x-4}{5} \lim_{n\to\infty} \cdot \frac{n^3}{(n+1)^3} = \frac{x-4}{5} \lim_{n\to\infty} \cdot \frac{n^3}{(n)^3} = \frac{x-4}{5}$$

Now setting $|\rho| < 1$, we get

$$|\frac{x-4}{5}| < 1 \quad ,$$

which means

$$|x-4| < 5 \quad .$$

We get that the **radius of convergence** is $R = 5$ and the **center of convergence** is 4. The **tentative interval of convergence** is $(4-5, 4+5) = (-1, 9)$.

We now have to **check the endpoints**. Plugging $x = -1$ into the infinite power series gives

$$\sum_{n=1}^{\infty} \frac{(-1-4)^n}{n^3 5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n^3 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \quad ,$$

which converges absolutely by the p-test. Plugging-in $x = 9$ gives

$$\sum_{n=1}^{\infty} \frac{(9-4)^n}{n^3 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n^3 5^n} = \sum_{n=1}^{\infty} \frac{1}{n^3} \quad ,$$

which converges (and hence converges absolutely). So we have to include **both** endpoints, and the **final interval of convergence** is $[-1, 9]$ (that can also be written as $-1 \le x \le 9$.

Ans. to 7: Radius of Convergence is 5, interval of convergence is $[-1, 9]$.

Note: In this question it was not asked about the **interval of absolute convergence**, but if it would have, then the interval of absolute convergence is also $[-1, 9]$ since the power series converges absolutely at both endpoints.

8. (10 points) Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{x^3}{32 + x^5} \quad .$$

Solution to 8: First write $f(x)$ as

$$f(x) = \frac{x^3}{32 + x^5} = \frac{x^3}{32(1 + x^5/32)} = \frac{x^3}{32} \cdot \frac{1}{1 + \frac{x^5}{32}}$$

$$\frac{x^3}{32} \cdot \frac{1}{1 - \frac{-x^5}{32}}$$

Now use the famous **geometrical series**

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad (|w| < 1)$$

with $w = \frac{-x^5}{32}$, to get

$$f(x) = \frac{x^3}{32} \cdot \sum_{n=0}^{\infty} (\frac{-x^5}{32})^n = \frac{x^3}{32} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{32^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+3}}{32^{n+1}} \quad ,$$

valid for $|\frac{-x^5}{32}| < 1$, which means $|x|^5 < 32$, which means $|x| < 2$.

Ans. to 8): The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+3}}{32^{n+1}} \quad ,$$

and the interval of convergence is $(-2, 2)$ (or $|x| < 2$).

Important!: Many people keep going and use the ratio test or root test to find the interval of convergence. This is a **waste of time**. All you have to do is decide on the right w and manipulate $|w| < 1$.

9 (10 points) Find the Maclaurin series for $f(x) = 3\cos 2x$ using the definition of a Maclaurin series.

Solution of 9.

Note: If you didn't have the phrase "from the definition" you should proceed as follows

$$\cos w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n} \quad ,$$

so replacing w by $2x$ and multiplying by 3 gives

$$3\cos(2x) = 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-2)^n \cdot 3}{(2n)!} x^{2n} \quad .$$

BUT, that's not how you were supposed to do it. Nevertheless, it is a good idea to do the above on scratch paper to know what to expect.

Since you were asked to do it **from the definition**, you are supposed to use

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad .$$

So write down $f(x)$ and a few of its derivatives

$$f(x) = 3\cos(2x) \quad f'(x) = 3 \cdot (-1)(2)\sin(2x) \quad f''(x) = 3 \cdot (-1)(2)^2 \cos(2x)$$

$$f'''(x) = 3 \cdot (-1)^2 (2)^3 \sin(2x)$$

$$f^{(4)}(x) = 3\cdot(-1)^2(2)^4 \cos(2x) \quad f^{(5)}(x) = 3\cdot(-1)^3(2)^5 \sin(2x) \quad f^{(6)}(x) = 3\cdot(-1)^3(2)^6 \cos(2x)$$

Plugging-in $x = 0$ we get

$$f(0) = 3 \quad f'(x) = 0 \quad f''(0) = 3 \cdot (-1)(2)^2 \quad f'''(0) = 0$$

$$f^{(4)}(0) = 3 \cdot (-1)^2(2)^4 \quad f^{(5)}(x) = 0 \quad f^{(6)}(x) = 3 \cdot (-1)^3(2)^6$$

We can **detect a pattern**: $f^{(2n+1)}(0) = 0$ and $f^{(2n)}(0) = 3(-1)^n 2^{2n}$. Putting it into the definitin of Maclaurin series, we see that the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{3(-1)^n 2^{2n}}{(2n)!} x^{2n} \quad .$$

10. (10 points) Find the first four non-zero terms of the Maclaurin expansion of

$$f(x) = e^{2x} \ln(1+3x)$$

Solution of 10: The first 4 non-zero terms of e^{2x} are

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots = 1 + 2x + 2x^2 + (4/3)x^3 + \ldots$$

The first four non-zero terms of $\ln(1+3x)$ are a bit harder since you don't know by heart ghe Maclaurin expansion of $\ln(1+w)$. Of course you can do it from the definition (like in # 9), but it is better to note that

$$[\ln(1+3x)]' = \frac{3}{1+3x}$$

So

$$\ln(1+3x) = \int \left(\frac{3}{1+3x}\right) dx$$

So we first find the Maclaurin expansion of

$$\frac{3}{1+3x} = 3 \cdot \frac{1}{1-(-3x)} = 3 \cdot (1 + (-3x) + (-3x)^2 + (-3x)^3) = 3 - 9x + 27x^2 - 81x^3 + \ldots$$

Now we use **term-by-term integration**

$$\ln(1+3x) = \int \left(\frac{3}{1+3x}\right) = C + \int (3 - 9x + 27x^2 - 81x^3 + \ldots)\, dx =$$

$$C + 3x - 9\frac{x^2}{2} + 27\frac{x^3}{3} - 81\frac{x^4}{4} + \ldots =$$

$$C + 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots \quad ,$$

when $x = 0$, $C = 0$ (since $\ln 1 = 0$). So we have

$$\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots \quad ,$$

This is only the **subproblem**. Combining the Maclaurin expansions of e^{2x} and $\ln(1+3x)$ and multiplying out, but **only keeping** terms up to x^4 (replace the rest by ...)

$$e^{2x} \ln(1+3x) =$$

$$(1+2x+2x^2+(4/3)x^3+\ldots)\cdot(3x-\frac{9}{2}x^2+9x^3-\frac{81}{4}x^4+\ldots) = 3x-\frac{9}{2}x^2+9x^3-\frac{81}{4}x^4+6x^2-9x^3+18x^4+6x^3-9x$$

$$3x + (3/2)x^2 + 6x^3 - (29/4)x^4 + \ldots$$

Ans. to 10: $3x + (3/2)x^2 + 6x^3 - (29/4)x^4 + \ldots$.