MATH 152 (07-09), Dr. Z. , **Solution to Second Midterm**

1. (10 points) Use the integral test to determine whether the series is convergent or divergent. Explain everything!

$$\sum_{n=1}^{\infty} \frac{n}{e^{2n}}$$

Solution of (1): The series has the same convergence status as the improper integral

$$\int_1^{\infty} \frac{x}{e^{2x}}\, dx \quad .$$

So let's try to evaluate it. First we need the indefinite integral. Writing the integral as

$$\int_1^{\infty} xe^{-2x}\, dx \quad ,$$

we use **integration by parts** with $u = x$, $v' = e^{-2x}$. This gives $u' = 1$ and $v = \frac{e^{-2x}}{-2} = (-1/2)e^{-2x}$. So

$$\int_1^{\infty} xe^{-2x}\, dx = x\frac{e^{-2x}}{-2} - \int 1 \cdot (-1/2)e^{-2x}\, dx = (-1/2)xe^{-2x} + \frac{1}{2}\int e^{-2x}\, dx =$$

$$(-1/2)xe^{-2x} + \frac{1}{2}\frac{e^{-2x}}{-2} = (-1/2)xe^{-2x} - (1/4)e^{-2x} = -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}}$$

Now, we put limits from 1 to R:

$$\int_1^{R} xe^{-2x}\, dx = -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}}\Big|_1^R = -\frac{R}{2e^{2R}} - \frac{1}{4e^{2R}} + \frac{1}{2e^2} + \frac{1}{4e^2}$$

Now we take the limit at $R \to \infty$, and get

$$\int_1^{\infty} xe^{-2x}\, dx = \lim_{R\to\infty}\left(-\frac{R}{2e^{2R}} - \frac{1}{4e^{2R}} + \frac{1}{2e^2} + \frac{1}{4e^2}\right)$$

The first limit is 0 by L'Hôpital and the second is $1/\infty = 0$, so the answer is a **finite** number, hence the improper integral converges, and by the integral test, the series converges.

Ans. to 1): Series converges by the integral test.

Important note: This is much easier with either the ratio test or root test ($\rho = 1/e^2 < 1$), so if you are not specifically requested to use the integral test use another test!

2. (10 points, 5 each) Determine whether the following series converge or diverge. Explain what test(s) you are using.

$$(a) \quad \sum_{n=1}^{\infty} \frac{7n^2+8n+4\sqrt{n}}{n^4+n+9} \quad ,$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{7+4\sqrt{n}}{n^{6/5}} \quad .$$

Solution to 2a): By the Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{7n^2+8n+4\sqrt{n}}{n^4+n+9}$$

has the same convergence status as

$$\sum_{n=1}^{\infty} \frac{7n^2}{n^4} = 7\sum_{n=1}^{\infty} \frac{1}{n^2} \quad ,$$

which converges by the p-test ($p=2$).

Ans. to 2a): Converges by Limit Comparison and p-test.

Solution to 2b): By the Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{7+4\sqrt{n}}{n^{6/5}}$$

has the same convergence status as the simpler series

$$\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^{6/5}} = 4\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^{6/5}} = 4\sum_{n=1}^{\infty} \frac{1}{n^{7/10}}$$

which diverges by the p-test ($p = 7/10 < 1$).

Ans. to 2b): Diverges by Limit Comparison and p-test.

3. (10 points: 3,3,4 resp.) Determine whether the following series converge or diverge

$$(a) \sum_{n=1}^{\infty} \frac{\sin^4 n}{n^2+1} \quad , \quad (b) \sum_{n=1}^{\infty} \frac{1}{n-10\sqrt{n}} \quad , \quad (c) \sum_{n=1}^{\infty} \frac{1}{5-2^{-n}} \quad .$$

Solution of 3a): The trig functions sin and cos are always between -1 and 1 so their square is less than 1. By the **(straight) comparison test**

$$\sum_{n=1}^{\infty} \frac{\sin^4 n}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Now by Limit (or straight) comparison test, and the p-test ($p = 2$) this is convergent, so

Ans. to 3a): Convergent by Comparison Test and p-test.

Solution of 3b): You can use (Straight-) Comparison but it is much easier to use Limit-Comparison.

$$\sum_{n=1}^{\infty} \frac{1}{n - 10\sqrt{n}}$$

has the same convergence status as

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad ,$$

which is the famous harmonic series (or p-series with $p = 1$) and so diverges.

Ans. to 3b): Divergent by Limit Comparison Test and p-test ($p = 1$).

Solution of 3c): By Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{5 - 2^{-n}}$$

has the same convergence status as (since 2^{-n} is tiny in the long run)

$$\sum_{n=1}^{\infty} \frac{1}{5} \quad ,$$

and this equals ∞, hence diverges.

COMMON MISTAKE: Many people 'forget' about the 5, and usually when 5 is next to n or n^2 they are right, but in this problem 5 is very significant since 2^{-n} is insignificant.

Another way: use the divergence test

$$\lim_{n\to} \frac{1}{5 - 2^{-n}} = \frac{1}{5}$$

which is not 0, so it diverges by the divergence test.

Ans. to 3c): Divergent by Limit Comparison Test and common sense (or p-test with $p = 0$). Or : Divergent by Divergence Test.

4. (10 points) Use the sum of the first 3 terms to approximate the sum of the series. Estimate the error.

$$\sum_{n=1}^{\infty} \frac{n+1}{(n+3)5^n} \quad .$$

Solution of 4: The first three terms are

$$\sum_{n=1}^{3} \frac{n+1}{(n+3)5^n} = \frac{1+1}{(1+3)\cdot 5^1} + \frac{2+1}{(2+3)\cdot 5^2} + \frac{3+1}{(3+3)\cdot 5^3} = \frac{1}{10} + \frac{3}{125} + \frac{2}{375} \quad .$$

The remainder is

$$\sum_{n=4}^{\infty} \frac{n+1}{(n+3)5^n} \le \sum_{n=4}^{\infty} \frac{1}{5^n}$$

since $(n+1)/(n+3) < 1$. The right side is a **geometric series**

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \frac{1}{5^4} + \frac{1}{5^5} + \frac{1}{5^6} + \ldots = \frac{1}{5^4} \cdot \left(\frac{1}{5} + \frac{1}{5^2} + \ldots \right) = \frac{1}{5^4} \cdot \frac{1}{1-1/5} = \frac{1}{5^4} \cdot \frac{5}{4} = \frac{1}{600}$$

(We used the geometric series sum

$$1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r} \quad ,$$

).

Ans. to 4) The approximation to the series using the first three terms is $\frac{1}{10} + \frac{3}{125} + \frac{2}{375}$ and the error is $\le 1/600$.

5. (10 points, 5 each) Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

$$(a) \quad \sum_{n=1}^{\infty} \frac{n^3}{2^n} \quad ,$$

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)} \quad ,$$

Solution of 5a): Use the ratio test

$$a_n = \frac{n^3}{2^n} \quad ,$$

$$a_{n+1} = \frac{(n+1)^3}{2^{n+1}} \quad ,$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^3}{2^{n+1}}}{\frac{n^3}{2^n}}$$

$$= \frac{(n+1)^3 2^n}{n^3 2^{n+1}} = \frac{(n+1)^3 2^n}{n^3 2^n \cdot 2^1} = \frac{(n+1)^3}{2n^3}$$

Now take the limit of the ratio

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{2n^3} = \lim_{n\to\infty} \frac{n^3}{2n^3} = \frac{1}{2} \quad .$$

Since $\rho = \frac{1}{2}$ is less than 1:

Ans. to 5a): Converges Absolutely by ratio test.

Solution of 5b): We first consider the **absolute version**

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{\ln(n+1)} = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

By the Limit Comparison Test this has the same convergence status as

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \quad ,$$

and this is the same as

$$\sum_{n=2}^{\infty} \frac{1}{n^0 (\ln n)^1} \quad ,$$

which diverges by Dr. Z's $p-q$ test ($p=0, q=1$). Since $p<1$ it diverges. Hence the absolute version diverges, and so the series is **not** absolutely convergent.

This leaves two options: cond. conv. and divergent. Since the original series is alternating, we try to use the Alternating Series Test with

$$b_n = \frac{1}{\ln(n+1)} \quad ,$$

This is (i) decreasing (since $\ln(x+1)$ is increasing its reciprocal is decreasing) (ii) tends to 0 (since $\ln(x+1)$ goes to ∞ its reciprocal goes to 0) It follows by the Alternating Series Test that the series is convergent. Since it is convergent but not absolutely convergent it is **condtionally convergent**.

Ans. to 5b): Conditionally Convergent.

6. (10 points, 5 each) Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

$$(a) \sum_{n=1}^{\infty} \frac{(-3)^n n!}{n^n} \quad .$$

$$(b) \sum_{n=1}^{\infty} \frac{2^n n^n}{n!} \quad .$$

Solution of 6a):

We use the ratio test

$$|a_n| = \frac{3^n n!}{n^n}$$

$$|a_{n+1}| = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}$$

First let's form the ratio and **simplify** as much as (legally!) possible

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{3^n n!}{n^n}}$$

which simplifies to

$$\frac{3^{n+1}n^n(n+1)!}{3^n n!(n+1)^{n+1}} = \frac{3^n \cdot 3n^n(n+1)!}{3^n n!(n+1)^{n+1}} = 3 \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)!}{n!}$$

$$= 3 \cdot \frac{n^n}{(n+1)^{n+1}}(n+1) = 3 \cdot \frac{n^n}{(n+1)^n} \quad .$$

This is the **simplified ratio**. Now take the limit as $n \to \infty$:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} 3 \cdot \frac{n^n}{(n+1)^n} = 3 \lim_{n\to\infty} \cdot \frac{n^n}{(n+1)^n} = \frac{3}{e} \quad .$$

Here we use the reciprocal of the famous formula in the formula sheet:

$$\lim_{n\to\infty} (\frac{n+1}{n})^n = e$$

(where $e = 2.71828...$)

Since $\rho = 3/e$ is bigger than 1 we get

Ans. to 6a): Series diverges.

Warning: My shortcut of "forgetting about the little ones" does not apply here.

$$\lim_{n\to\infty}(\frac{n+1}{n})^3 = \lim_{n\to\infty}(\frac{n}{n})^3 = 1$$

and

$$\lim_{n\to\infty}(\frac{n+1}{n})^{1000} = \lim_{n\to\infty}(\frac{n}{n})^{1000} = 1$$

But if the exponent is n then you can **NOT** do it.

Solution of 6b):

We use the ratio test

$$|a_n| = \frac{2^n n^n}{n!}$$

$$|a_{n+1}| = \frac{2^{n+1}(n+1)^{n+1}}{n!}$$

First let's form the ratio and **simplify** as much as (legally!) possible

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}(n+1)^{n+1}}{(n+1)!}}{\frac{2^n n^n}{n!}}$$

which simplifies to

$$\frac{2^{n+1}(n+1)^{n+1}n!}{2^n(n+1)!n^n} = 2\cdot\frac{(n+1)^n}{n^n}$$

This is the **simplified ratio**. Now take the limit as $n\to\infty$:

$$\rho = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} 2\cdot\frac{(n+1)^n}{n^n} 2\lim_{n\to\infty}\cdot\frac{(n+1)^n}{n^n} = 2e$$

Here we use the famous formula in the formula sheet:

$$\lim_{n\to\infty}(\frac{n+1}{n})^n = e$$

(where $e = 2.71828...$)

Since $\rho = 2e$ is bigger than 1 we get

Ans. to 6b): Series diverges.

7. (10 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(x+1)^n}{n2^n} \quad .$$

Sketch of the Solution of 7: The limit of the ratios is

$$\rho = \frac{x+1}{2}$$

(do it!) Setting $|\rho| < 1$ gives

$$|\frac{x+1}{2}| < 1$$

which is the same as (multiply both sides by 2) as

$$|x+1| < 2$$

that gives: **radius of convergence** is 2 and **center of convergence** is $x = -1$ so the **tentative interval of convergence** is $(-1-2, -1+2) = (-3, 1)$.

Now we have to check the endpoints. When $x = -3$ the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is convergent by the Alternating Series Test. So we add $x = -3$. When $x = 1$ the power series becomes

$$\sum_{n=1}^{\infty} \frac{(1+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad ,$$

which diverges by the p-test ($p = 1$), so we **don't** add $x = 1$ to the interval of convergence.

Ans. to 7: Radius of convergence is 2 and interval of convergence is $[-3, 1)$ that can be written as $-3 \leq x < 1$.

8. (10 points) Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{x^2}{1+27x^3} \quad .$$

Solution of 8:

$$f(x) = \frac{x^2}{1+27x^3} = x^2 \cdot \frac{1}{1-(-27x^3)} \quad .$$

We use the **geometric series**

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad , (|w| < 1)$$

with $w = -27x^3$.

$$f(x) = x^2 \cdot \frac{1}{1-(-27x^3)} = x^2 \cdot \sum_{n=0}^{\infty}(-27x^3)^n =$$

$$x^2 \cdot \sum_{n=0}^{\infty}(-27)^n(x^3)^n = x^2 \cdot \sum_{n=0}^{\infty}(-27)^n x^{3n} = \sum_{n=0}^{\infty}(-27)^n x^{3n+2}$$

This is valid for $|-27x^3| < 1$ which is the same as $|x|^3 < 1/27$ which is the same as $|x| < 1/3$. So the **interval of convergence** is $(-1/3, 1/3)$.

Ans. to 8): The Maclaurin series of $f(x)$ is $\sum_{n=0}^{\infty}(-27)^n x^{3n+2}$ and the interval of convergence is $(-1/3, 1/3)$.

VERY IMPORTANT: Some people find the interval of convergence the long way, doing the ratio test etc. This is a **waste** of time. Whenever you use the geometric series for $1/(1-w)$ it is automatic that it is valid for $|w| < 1$ so all you have to do is solve the inequality $|w| < 1$, like I did above.

9. (10 points) Find the Maclaurin series for $f(x) = 4e^{x+1}$ using the definition of a Maclaurin series.

Solution of 9):

It is tempting to first simplify

$$f(x) = 4e^x \cdot e^1 = (4e)e^x \quad ,$$

and then use the memorized Maclaurin series for e^x getting that the Maclaurin series for $f(x)$ is

$$4e\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{4e}{n!}x^n$$

and if the question didn't mention "from the definition" it would have been OK. But since it did we have to do it the long way. First write down $f(x)$ and its derivatives

$$f(x) = 4e^{x+1} \quad f'(x) = 4e^{x+1} \quad f''(x) = 4e^{x+1} \quad f'''(x) = 4e^{x+1}$$

and you see that they are all the same, so $f^{(n)}(x) = 4e^{x+1}$ Plugging in $x = 0$ we get

$$f(0) = 4e^{0+1} = 4e \quad f'(0) = 4e^{0+1} = 4e \quad f''(0) = 4e^{0+1} = 4e \quad f'''(0) = 4e^{0+1} = 4e$$

and in general $f^{(n)}(0) = 4e^{0+1} = 4e$. Using the **definition**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

we get

$$f(x) = \sum_{n=0}^{\infty} \frac{4e}{n!} x^n \quad .$$

Common Mistake: Many people plugged-in $w = x + 1$ in

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

getting

$$f(x) = \sum_{n=0}^{\infty} \frac{4(x+1)^n}{n!}$$

This is the **right answer to the wrong question**, since this is a Taylor series centered at $x = -1$ and not a Maclaurin series (that is centered at $x = 0$).

10. (10 points, 5 each) (a) Expand $\sqrt[3]{1+x}$ as a power series. (b) Use part (a) to estimate $\sqrt[3]{1.01}$ correct to four decimal

Solution of (a): We use the **Binomial Series** from the Formula Sheet

$$(1+x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

or in the ... notation

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2} w^x + \ldots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n + \ldots$$

Here $k = 1/3$ so

Ans. to a)

$$(1+x)^{1/3} = 1 + \sum_{n=1}^{\infty} \frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!} x^n \quad ,$$

or even more compactly

$$(1+x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^n \quad .$$

Solution of b): We plug-in $x = .01$ and $k = 1/3$ into the ... form and write down the first few terms

$$(1{+}0.01)^{1/3} = 1{+}(1/3)(0.01){+}\frac{(1/3)(1/3-1)}{2}(0.01)^2{+}\frac{(1/3)(1/3-1)(1/3-2)}{6}(0.01)^3{+}\ldots{+} \quad .$$

This is an alternating series and we quit as soon as the terms are less the prescribed error (four decimals means that the error is $5 \cdot 10^{-5}$. Already the third term is less than that, so we only keep the first two terms and the approximation is

$$(1 + 0.01)^{1/3} \approx 1 + (1/3)(0.01) = 1.0033$$

Ans. to 10b): 1.0033.