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MATH 251 (1-3), Dr. Z. , FINAL, 8:00-11:00am , Friday, Dec. 15, 2006 [Blue Version]

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1. (13 pts.) Find the curvature of the curve $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ at the point $(1, 1, \frac{2}{3})$.

Solution:

$$\mathbf{r}'(t) = \langle 1, 2t, 2t^2 \rangle \quad ,$$

$$\mathbf{r}''(t) = \langle 0, 2, 4t \rangle \quad .$$

Since we are interested at the point $(1, 1, \frac{2}{3})$, setting this equal to $\langle t, t^2, \frac{2}{3}t^3 \rangle$, gives $t = 1$. **Now** plug-in $t = 1$, to get

$$\mathbf{r}'(1) = \langle 1, 2, 2 \rangle \quad ,$$

$$\mathbf{r}''(1) = \langle 0, 2, 4 \rangle \quad .$$

The formula for the curvature is:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad .$$

So

$$\kappa(1) = \frac{|\langle 1, 2, 2 \rangle \times \langle 0, 2, 4 \rangle|}{|\langle 1, 2, 2 \rangle|^3} \quad .$$

First we compute the cross the product:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} =$$

$$\mathbf{i}\begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= \mathbf{i}(2 \cdot 4 - 2 \cdot 2) - \mathbf{j}(1 \cdot 4 - 0 \cdot 2) + \mathbf{k}(1 \cdot 2 - 2 \cdot 0) =$$

$$4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = \langle 4, -4, 2 \rangle \quad .$$

So

$$\kappa(1) = \frac{|\langle 4, -4, 2 \rangle|}{|\langle 1, 2, 2 \rangle|^3} =$$

$$\frac{\sqrt{4^2 + (-4)^2 + 2^2}}{(\sqrt{1^2 + 2^2 + 2^2})^3} = \frac{\sqrt{36}}{(\sqrt{9})^3} = \frac{6}{3^3} = \frac{2}{9} \quad .$$

Ans.: $\frac{2}{9}$.

2. (13 points) By using Stokes's Theorem, or otherwise, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$F(x,y,z) = yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k} \quad ,$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above. Be sure to explain everything.

Solution: Stokes theorem says that

$$\int_C \mathbf{F} \cdot d\,\mathbf{r} = \int\int curl\,\mathbf{F} \cdot d\mathbf{S} \quad ,$$

where S is *any* open surface whose boundary is C. But $curl\mathbf{F} = \langle 0, 0, 0 \rangle$ (you do it!), so it *does not matter* what S is, since the surface-integral of 0 is always 0.

Ans. 0.

3. (13 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^3 + y^3 + z^3 = 5xyz + 1 \quad .$$

Solution: The easiest way is to use the formulas for implicit differentiation. If $F(x, y, z) = 0$ then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \quad .$$

First we make

$$x^3 + y^3 + z^3 = 5xyz + 1$$

into

$$x^3 + y^3 + z^3 - 5xyz - 1 = 0 \quad ,$$

so in this problem

$$F(x, y, z) = x^3 + y^3 + z^3 - 5xyz - 1 \quad .$$

Now $F_x = 3x^2 - 5yz$, $F_y = 3y^2 - 5xz$, $F_z = 3z^2 - 5xy$, and we have

$$\frac{\partial z}{\partial x} = -\frac{3x^2 - 5yz}{3z^2 - 5xy} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{3y^2 - 5xz}{3z^2 - 5xy} \quad .$$

These are the **answers**.

4. (13 points) Find an equation for the tangent plane to the parametric surface:

$$x = u^2 \quad , \quad y = u + v \quad , \quad z = v^2 \quad ,$$

at the point $(1, 2, 1)$.

Solution: This is a **parametric surface**. Writing it in **vector form** we have:

$$\mathbf{r}(u, v) = \langle\, u^2 \,,\, u + v \,,\, v^2 \rangle.$$

Taking partial derivatives with respect to u and v we have

$$\mathbf{r}_u = \langle\, 2u \,,\, 1 \,,\, 0 \rangle \quad ,$$

$$\mathbf{r}_v = \langle\, 0 \,,\, 1 \,,\, 2v \rangle \quad .$$

What are the actual values of u and v at our point $(1, 2, 1)$?. Solving $1 = u^2, 2 = u + v, 1 = v^2$ gives $u = 1$ and $v = 1$. Plugging-in $u = 1$ and $v = 1$ above we get

$$\mathbf{r}_u(1, 1) = \langle\, 2 \,,\, 1 \,,\, 0 \rangle \quad ,$$

$$\mathbf{r}_v(1, 1) = \langle\, 0 \,,\, 1 \,,\, 2 \rangle \quad .$$

Now a **normal vector** is $\mathbf{r}_u \times \mathbf{r}_v$. Taking cross-product

$$\langle\, 2 \,,\, 1 \,,\, 0 \rangle \times \langle\, 0 \,,\, 1 \,,\, 2 \rangle = \langle\, 2 \,,\, -4 \,,\, 2 \rangle \quad .$$

The equation of plane with normal vector $\langle a, b, c \rangle$ passing through a point (x_0, y_0, z_0) is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. So the equation of the tangent plane is:

$$2(x - 1) - 4(y - 2) + 2(z - 1) = 0 \quad .$$

Dividing by 2 gives:

$$(x - 1) - 2(y - 2) + (z - 1) = 0 \quad .$$

Simplifying, gives: $x - 1 - 2y + 4 + z - 1 = 0$, which is $x - 2y + z + 2 = 0$, or $z = -x + 2y - 2$.

Ans.: $x - 2y + z + 2 = 0$, or $z = -x + 2y - 2$.

5. (13 points) Change the order of integration in

$$\int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx \quad .$$

Solution: This iterated integral, of type-I, can be written as the **double-integral** $\int\int_D f(x,y)\, dy\, dx$, where D is the region

$$D = \{(x,y)\,|\, 1 \le x \le 2\,,\, 0 \le y \le \ln x\,\}.$$

This is a "triangle-like" region with vertices $(1,0)$, $(2,0)$, and $(2, \ln 2)$. We have to express it as a type-II region. Its projection on the y-axis is the interval $0 < y < \ln 2$. This is the "main-road". A typical horizontal cross-section starts at the curve $y = \ln x$, which we now write as $x = e^y$ and ends at the vertical line $x = 2$, so the region D, written in type-II style is

$$D = \{(x,y)\,|\, 0 \le y \le \ln 2\,,\, e^y \le x \le 2\,\},$$

and our double-integral in $dx\,dy$-type is

$$\int_0^{\ln 2} \int_{e^y}^2 f(x,y)\, dx\, dy \quad .$$

Ans.: $\int_0^{\ln 2} \int_{e^y}^2 f(x,y)\, dx\, dy \quad .$

6. (13 points) Let

$$\mathbf{F}(x,y,z) =$$

$$\langle \cos(\sqrt{1+x}+x^3)\,,\,\tan((1+\cos(\sqrt{1+x}+x^3))^7)\,,\,\tan^{-1}((e^{x^2}+\cos(\sqrt{1+x}+x^3))^7\rangle$$

and let $\langle\, P\,,\, Q\,,\, R\,\rangle = curl\,\mathbf{F}$. Compute

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z} \quad .$$

Be sure to explain everything.

Solution: $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$ is $div\langle P, Q, R\rangle$. Since $\langle\, P\,,\, Q\,,\, R\,\rangle = curl\,\mathbf{F}$, what we need is $div\langle\, P\,,\, Q\,,\, R\,\rangle = div\, curl\,\mathbf{F}$. But $div\, curl\,\mathbf{F}$ is **always** 0 no matter what $\mathbf{F}$ is, so the answer is 0.

Ans.: 0.

7. (13 points) Let C be the line segment from $(0,1)$ to $(3,5)$, find $\int_C 2xy\, ds$.

Solution: The **parametric equation** for the line-segment joining $(0,1)$ and $(3,5)$ is

$$\mathbf{r}(t) = (1-t)\langle 0,1\rangle + t\langle 3,5\rangle = \langle 3t, 1+4t\rangle \quad .$$

So $x = 3t$ and $y = 1+4t$. We have $ds = |\mathbf{r}'(t)|dt$. Since $\mathbf{r}'(t) = \langle 3,4\rangle$, $ds = |\langle 3,4\rangle|dt = \sqrt{3^2+4^2}dt = 5dt$.

How line-integral becomes the definite integral

$$\int_C 2xy\, ds = \int_0^1 2(3t)(1+4t)\,5\, dt = 30\int_0^1 t(1+4t)\, dt = 30\int_0^1 t + 4t^2\, dt =$$
$$30\left(\frac{t^2}{2} + \frac{4t^3}{3}\right)\Big|_0^1 = 30\left(\frac{1}{2} + \frac{4}{3} - 0\right) = 30\cdot\frac{11}{6} = 55 \quad .$$

Ans.: 55.

8. (13 points) Evaluate

$$\int_C (5y - \sin(e^x))\, dx + (10x - e^{\cos^2 y})\, dy \quad ,$$

where C is the closed curve consisting of the boundary of the rectangle

$$\{\, (x, y) \,|\, 0 \leq x \leq 4\,,\, 0 \leq y \leq 3 \,\} \quad .$$

Solution: Use **Green's Theorem**:

$$\int_C P\, dx + Q\, dy = \int\int_D (Q_x - P_y)\, dA \quad ,$$

where D is the region that is **inside** C. In our case D is the above rectangle.

In this problem $P = 5y - sin(e^x)$ and $Q = 10x - e^{\cos^2 y}$. $Q_x = 10$ and $P_y = 5$, so $Q_x - P_y = 5$ and the desired answer is $\int\int_D 5\, dA$, that can be done directly, but more efficiently equals $5 \int\int_D dA$ which is 5 times the area of the rectangle, which is $3 \cdot 4 = 12$. So the answer is $5 \cdot 12 = 60$.

Ans.: 60.

9. (12 points) Find the Jacobian of the transformation

$$x = u + v + w \quad , \quad y = u^2 + v^2 + w^2 \quad , \quad z = u^3 + v^3 + w^3 \quad .$$

Solution: The Jacobian is:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2u & 2v & 2w \\ 3u^2 & 3v^2 & 3w^2 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 2v & 2w \\ 3v^2 & 3w^2 \end{vmatrix} \quad - 1 \cdot \begin{vmatrix} 2u & 2w \\ 3u^2 & 3w^2 \end{vmatrix} \quad + 1 \cdot \begin{vmatrix} 2u & 2v \\ 3u^2 & 3v^2 \end{vmatrix}$$

$$= [(2v)(3w^2) - (2w)(3v^2)] - [(2u)(3w^2) - (2w)(3u^2)] + [(2u)(3v^2) - (2v)(3u^2)]$$

$$= 6(uv^2 - uw^2 - vu^2 + vw^2 + wu^2 - wv^2) \quad .$$

Ans.: $6(uv^2 - uw^2 - vu^2 + vw^2 + wu^2 - wv^2)$.

Remark: Using algebra, this can be factored to be $6(v-u)(w-u)(w-v)$.

10. (12 points) Set-up but do not evaluate, a triple iterated integral for the volume of the solid bounded by the cylinder $y = x^2$ and planes $z = 0$ and $y + z = 1$.

Solution: The surface $y + z = 1$ (that happens to be a plane) can be written, explicitly, as $z = 1 - y$. To see where it meets the xy-plane we set $z = 0$ and get $1 - y = 0$, which is $y = 1$. So the projection of our 3D region on the xy-plane is the region bounded by $y = x^2$ and $y = 1$ (draw it). The full region can be written as

$$E = \{(x, y, z) \mid -1 \leq x \leq 1\,,\; x^2 \leq y \leq 1\,,\; 0 \leq z \leq 1 - y\} \quad .$$

The volume is the **volume-integral** $\int\int\int_E 1\, dV$, and writing it as an iterated integral gives

$$\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} dz\, dy\, dx \quad .$$

This is the **answer**.

11. (12 points) Use Lagrange multipliers (no credit for other methods!) to find the largest value that xyz can be, given that $x + y + z = 3$.

Solution: $f = xyz$, $g = x + y + z - 3$. $\bigtriangledown f = \langle yz, xz, xy \rangle$, $\bigtriangledown g = \langle 1, 1, 1 \rangle$. Setting $\bigtriangledown f = \lambda \bigtriangledown g$, we get $\langle yz, xz, xy \rangle = \langle \lambda, \lambda, \lambda \rangle$. We have to solve the system of four equations and four unknowns.

$$yz = \lambda \quad , \quad xz = \lambda \quad , \quad xy = \lambda \quad , \quad , x + y + z = 3 \quad .$$

One possibility is $\lambda = 0$ then we get the solutions

$$\lambda = 0 \quad , \quad x = 0 \quad , \quad y = 0 \quad , \quad z = 3 \quad ,$$

$$\lambda = 0 \quad , \quad x = 0 \quad , \quad y = 3 \quad , \quad z = 0 \quad ,$$

$$\lambda = 0 \quad , \quad x = 0 \quad , \quad y = 0 \quad , \quad z = 3 \quad .$$

If $\lambda \neq 0$, then none of x, y, z are 0, and we can divide the second equation by the first, getting $y = x$, and the third equation by the second getting, $y = z$, so $x = y = z$, and plugging-in into the last equation gives $3x = 3$ which means that $x = 1$, and so $y = 1$ and $z = 1$.

The finalists are $(x, y, z) = (3, 0, 0)$, $(x, y, z) = (0, 3, 0)$, $(x, y, z) = (0, 0, 3)$, and $(x, y, z) = (1, 1, 1)$. Plugging-into $f(x, y, z) = xyz$, we get $f(3, 0, 0) = 0$, $f(0, 3, 0) = 0$,$f(0, 0, 3) = 0$, and $f(1, 1, 1) = 1$. The biggest among these is 1, so this is the **maximum value**.

Ans.: 1.

12. (12 points) Find an equation of the tangent plane to the surface $z = e^{2x-3y}$ at the point $(3, 2, 1)$.

Solution: Here the surface is given **explicitly** $z = f(x, y)$, where $f(x, y) = e^{2x-3y}$. The relevant formula is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

But first let's make sure that the problem makes sense, i.e. that the point $(3, 2, 1)$ lies on our surface. $f(3, 2) = e^{2(3)-3(2)} = e^0 = 1$, so it is OK.

No $f_x(x, y) = 2e^{2x-3y}$, $f_y(x, y) = -3e^{2x-3y}$, so $f_x(3, 2) = 2$, $f_y(3, 2) = -3$, and since $x_0 = 3, y_0 = 2, z_0 = 1$, the equation of the tangent plane is:

$$z - 1 = 2(x - 3) - 3(y - 2) = 2x - 3y \quad .$$

Simplifying, we get $z = 2x - 3y + 1$.

Ans.: $z = 2x - 3y + 1$.

13. (12 points) Find the local maximum and minimum points, the local maximum and minimum values, and saddle point(s) of the function $f(x,y) = 4x^2 + y^2 + 2x^2y - 1$.

Solution:

$$f_x = 8x + 4xy \quad , \quad f_y = 2y + 2x^2 \quad .$$

For future reference,

$$f_{xx} = 8 + 4y \quad , \quad f_{xy} = 4x \quad , \quad f_{yy} = 2 \quad .$$

We have to solve $f_x = 0, f_y = 0$, in other words the system

$$8x + 4xy = 0 \quad , \quad 2y + 2x^2 = 0 \quad .$$

From the second equation we get $y = -x^2$ **always**. Plugging this into the first equation gives $4x(2+y) = 4x(2-x^2) = 0$, whose solutions are $x = 0$, $x = -\sqrt{2}$, $x = \sqrt{2}$. Using $y = -x^2$, we have **three** solutions:

$$x = 0 \quad , \quad y = 0 \quad ;$$

$$x = \sqrt{2} \quad , \quad y = -2 \quad ;$$

$$x = -\sqrt{2} \quad , \quad y = -2 \quad .$$

These yield the three **finalists** $(0,0)$, $(\sqrt{2},-2)$, $(-\sqrt{2},-2)$.

At $(0,0)$,

$$f_{xx} = 8 \quad , \quad f_{xy} = 0 \quad , \quad f_{yy} = 2 \quad ,$$

so $D = 8 \cdot 2 - 0^2 = 16 > 0$. Since $f_{xx} = 8 > 0$ this is a **local minimum point**, and the local minimum value is $f(0,0) = -1$.

At $(-\sqrt{2},-2)$,

$$f_{xx} = 0 \quad , \quad f_{xy} = -4\sqrt{2} \quad , \quad f_{yy} = 2 \quad ,$$

and $D = 0 \cdot 2 - (-4\sqrt{2})^2 = -32 < 0$ so this is a **saddle point**.

At $(\sqrt{2},-2)$,

$$f_{xx} = 0 \quad , \quad f_{xy} = 4\sqrt{2} \quad , \quad f_{yy} = 2 \quad ,$$

and $D = 0 \cdot 2 - (4\sqrt{2})^2 = -32 < 0$ so this is a **saddle point**.

Answers:

Local minimum point: $(0,0)$; Local minimum value: -1.

Local maximum points: none ; Local maximum value: none.

Saddle points :$(-\sqrt{2},-2)$ and $(\sqrt{2},-2)$.

14. (12 points) Find the velocity and position vectors of a particle whose acceleration is $\mathbf{a}(t) = \mathbf{i} + \mathbf{j}$, and at $t = 0$ the velocity is $\mathbf{i} - \mathbf{j}$ and the position is $\mathbf{k}$.

Solution:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int (\mathbf{i} + \mathbf{j})\, dt = t\mathbf{i} + t\mathbf{j} + \mathbf{C}$$

But $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Plugging-in $t = 0$ above gives $\mathbf{C} = \mathbf{i} - \mathbf{j}$. Plugging this value of $\mathbf{C}$ back gives

$$\mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + \mathbf{i} - \mathbf{j} = (t+1)\mathbf{i} + (t-1)\mathbf{j} \quad .$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int ((t+1)\mathbf{i} + (t-1)\mathbf{j})\, dt = (\frac{t^2}{2} + t)\mathbf{i} + (\frac{t^2}{2} - t)\mathbf{j} + \mathbf{C} \quad .$$

Plugging-in $t = 0$ gives $\mathbf{r}(0) = \mathbf{C}$. By the problem $\mathbf{r}(0) = \mathbf{k}$, so $\mathbf{C} = \mathbf{k}$. Using this value of C we get

$$\mathbf{r}(t) = (\frac{t^2}{2} + t)\mathbf{i} \, + \, (\frac{t^2}{2} - t)\mathbf{j} + \mathbf{k} \quad .$$

Answers: The velocity vector is $(t+1)\mathbf{i} + (t-1)\mathbf{j}$ or $\langle t+1, t-1, 0 \rangle$.

The position vector is $(\frac{t^2}{2} + t)\mathbf{i} + (\frac{t^2}{2} - t)\mathbf{j} + \mathbf{k}$ or $\langle \frac{t^2}{2} + t, \frac{t^2}{2} - t, 1 \rangle$.

15. (12 points) Find an equation for the plane through the point $(1, 0, 2)$ that contains the line

$$\mathbf{r}(t) = \langle 1, 1, 1\rangle + t\langle 1, -1, 0\rangle \quad .$$

Solutions: We need two directions **along the plane**. One is obviously the direction vector of our line $\mathbf{a} = \langle 1, -1, 0\rangle$. Another direction can be gotten by taking any point on our line, for example when $t = 0$, it is $(1, 1, 1)$, and computing the direction vector between $(1, 0, 2)$ to $(1, 1, 1)$ that is $\mathbf{b} = \langle 1 - 1, 1 - 0, 1 - 2\rangle = \langle 0, 1, -1\rangle$. Taking the **cross-product** of $\mathbf{a}, \mathbf{b}$ gives

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1, -1, 0\rangle \times \langle 0, 1, -1\rangle = \langle 1, 1, 1\rangle \quad .$$

So $\mathbf{n} = \langle a, b, c\rangle = \langle 1, 1, 1\rangle$. Finally, we use the formula $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ with (x_0, y_0, z_0) being *any* point on our plane, for example, $(1, 0, 2)$ and we get $1 \cdot (x - 1) + 1 \cdot (y - 0) + 1 \cdot (z - 2) = 0$, that simplifies to $x + y + z = 3$.

Ans.: $x + y + z = 3$.

16. (12 points) Compute the limit

$$\lim_{(x,y,z)\to(1,1,1)} e^{-xy}\sin(\pi z/2)$$

or prove that it does not exist.

Solution: The first thing to do is **plug-it-in**. If everything makes sense (you are not dividing by 0) then the limit exists and equals to whatever you get!

$$\lim_{(x,y,z)\to(1,1,1)} e^{-xy}\sin(\pi z/2) = e^{-1\cdot 1}\sin(\pi/2) = e^{-1}\cdot 1 = \frac{1}{e} \quad .$$

Ans.: $\frac{1}{e}$.