NAME: (print!) Dr. Z.

Section: 4-6 **E-Mail address:** zeilberg at math dot rutgers dot edu

MATH 251 (4-6), Dr. Z. , FINAL, noon-3:00pm , Thurs., Dec. 21, 2006 [Blue Version]

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM
Do not write below this line

---

1. (out of 13) 13

2. (out of 13) 13

3. (out of 13) 13

4. (out of 13) 13

5. (out of 13) 13

6. (out of 13) 13

7. (out of 13) 13

8. (out of 13) 13

9. (out of 12) 12

10. (out of 12) 12

11. (out of 12) 12

12. (out of 12) 12

13. (out of 12) 12

14. (out of 12) 12

15. (out of 12) 12

16. (out of 12) 12

1. (13 points) Evaluate

$$\int_0^4 \int_{y/4}^1 \frac{12}{(x^2+1)^4}\, dx\, dy \quad ,$$

by inverting the order of integration and evaluating the new iterated integral.

Solution: This type-II iterated integral can be written as a **double-integral**

$$\int\int_D \frac{12}{(x^2+1)^4}\, dA \quad ,$$

where D is the type-II region

$$D = \{(x,y)\,|\,0 \leq y \leq 4\,,\, y/4 \leq x \leq 1\} \quad .$$

This is a triangle whose vertices are $(0,0)$, $(1,0)$ and $(1,4)$. Since $x = y/4$ is the same as $y = 4x$, this same region can be written in type-I style

$$D = \{(x,y)\,|\,0 \leq x \leq 1\,,\, 0 \leq y \leq 4x\} \quad ,$$

and the double-integral can be written as the type-I iterated integral

$$\int_0^1 \int_0^{4x} \frac{12}{(x^2+1)^4}\, dy\, dx \quad .$$

The **inner-integral** is

$$\int_0^{4x} \frac{12}{(x^2+1)^4}\, dy = \frac{12}{(x^2+1)^4}\int_0^{4x} dy = \frac{12}{(x^2+1)^4}(4x-0) = \frac{48x}{(x^2+1)^4} \quad .$$

The **outside-integral** is

$$\int_0^1 \frac{48x}{(x^2+1)^4}\, dx \quad .$$

Doing the **subsitution** $u = x^2+1$ we get $du = 2x\,dx$ so $dx = du/(2x)$. Also when $x=0$, $u=1$ and when $x=1$, $u=2$, so our integral is

$$\int_1^2 \frac{48x}{(u)^4}\frac{du}{2x} = \int_1^2 \frac{24}{(u)^4}\, du = \int_1^2 24u^{-4}\, du == 24\frac{u^{-3}}{-3}\Big|_1^2$$

$$= \frac{-8}{u^3}\Big|_1^2 = \frac{-8}{2^3} - \frac{-8}{1^3} = -1+8 = 7 \quad .$$

Ans.: 7.

2. Suppose that the **position** of a certain particle is given by

$$\mathbf{r}(t) = \langle e^t \cos t \, , \, e^t \sin t \, , \, e^t \rangle \quad , 0 \le t \le \pi \quad .$$

(a) (4 points) Find the **velocity** of the particle as a function of the time t.

(b) (9 points) Find the **length of the arc** traversed by the moving particle for $0 \le t \le \pi$.

Solution of a):

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle e^t (\cos t - \sin t) \, , \, e^t (\sin t + \cos t) \, , \, e^t \rangle$$

This is the **ans. to (a)**.

Solution of b):

$$|\mathbf{r}'(t)| = e^t |\langle (\cos t - \sin t) \, , \, (\sin t + \cos t) \, , \, \rangle| = e^t \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1^2}$$

$$= e^t \sqrt{\cos^2 t - 2 \sin t \cos t + \sin^2 t + \cos^2 t + 2 \sin t \cos t + \sin^2 t + 1^2}$$

$$= e^t \sqrt{\cos^2 t + \sin^2 t + \cos^2 t + \sin^2 t + 1^2} = e^t \sqrt{3} \quad .$$

The arclength is

$$\int_0^\pi ds = \int_0^\pi |\mathbf{r}'(t)| \, dt = \int_0^\pi \sqrt{3} e^t \, dt = \sqrt{3} e^t \Big|_0^\pi = \sqrt{3}(e^\pi - e^0) = \sqrt{3}(e^\pi - 1) \quad .$$

Ans. to (b): $\sqrt{3}(e^\pi - 1)$.

3. (13 points) Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$ as **functions of** r **and** s , if

$$f(x, y) = x^3 + 2xy + y^3 \quad ,$$

and the variables are related by $x = r - s$ and $y = r + s$. You do not need to simplify!

Solution: By the **chain-rule**

$$f_r = (f_x)(x_r) + (f_y)(y_r) \quad ,$$

$$f_s = (f_x)(x_s) + (f_y)(y_s) \quad .$$

Here $x_r = 1, x_s = -1, y_r = 1, y_s = 1,\ f_x = 3x^2 + 2y,\ f_y = 2x + 3y^2$. So

$$f_r = (3x^2 + 2y)(1) + (2x + 3y^2)(1) = 3x^2 + 3y^2 + 2x + 2y \quad ,$$

$$f_s = (3x^2 + 2y)(-1) + (2x + 3y^2)(1) = -3x^2 + 3y^2 + 2x - 2y \quad ,$$

Finally, expressing everything in terms of (r, s), we plug-in $x = r - s$ and $y = r + s$ to get

$$f_r = 3(r - s)^2 + 3(r + s)^2 + 2(r - s) + 2(r + s) \quad ,$$

$$f_s = -3(r - s)^2 + 3(r + s)^2 + 2(r - s) - 2(r + s) \quad .$$

The above is acceptable, since you weren't ask to simplify, but if you did you would get:

$$f_r = 6r^2 + 6s^2 + 4r \quad ,$$

$$f_s = 12rs - 4s \quad .$$

Remark: In this problem you don't need to use the chain-rule. We have

$$f = (r - s)^3 + 2(r - s)(r + s) + (r + s)^3 =$$

$$r^3 - 3r^2s + 3rs^2 - s^3 + 2r^2 - 2s^2 + r^3 + 3r^2s + 3rs^2 + s^3 = 2r^3 + 6rs^2 + 2r^2 - 2s^2 \quad .$$

Now we can do it **directly**: $f_r = 6r^2 + 6s^2 + 4r$ and $f_s = 12rs - 4s$. Getting the **same answer**.

4. Let

$$f(x, y, z) = -x^2 + y^2 + z^2 - 1 \quad .$$

(a) (2 points) Compute $\bigtriangledown f$.

(b) (5 points) Find a normal to the level surface $f(x, y, z) = 0$ at the point $(1, 1, 1)$, and give an equation for the tangent plane to that surface at that point.

(c) (6 points) Compute the directional derivative of $f(x, y, z)$ at the point $(1, 1, 1)$ in the direction $\langle\, 1\,,\, 2\,,\, 2\,\rangle$.

Sol. of (a): $\bigtriangledown f = \langle -2x, 2y, 2z \rangle$.

Sol. of (b): A normal direction at that point is $\bigtriangledown f(1, 1, 1) = \langle -2, 2, 2 \rangle$. An equation for the normal line is $\langle 1, 1, 1 \rangle + t\langle -2, 2, 2 \rangle$.

An equation for the normal plane is

$$(-2)(x - 1) + 2(y - 1) + 2(z - 1) = 0 \quad ,$$

that simplifies to $-x + y + z = 1$ or $z = 1 + x - y$.

Sol. of (c): Since $|\langle 1, 2, 2 \rangle| = \sqrt{1^2 + 2^2 + 2^2} = 3$, the unit vector in the direction of $\langle 1, 2, 2 \rangle$ is $\mathbf{u} = \langle 1/3, 2/3, 2/3 \rangle$. The directional derivative $D_{\mathbf{u}} f$ at $(1, 1, 1)$ is $\langle 1/3, 2/3, 2/3 \rangle \cdot \langle -2, 2, 2 \rangle = 2$.

Ans. to (c): 2.

5. (13 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$\sin(x + 2y + 3z) = 5xyz + 1 \quad .$$

Solution: First write the relationship as

$$\sin(x + 2y + 3z) - 5xyz - 1 = 0 \quad .$$

Here $F(x, y, z) = \sin(x + 2y + 3z) - 5xyz - 1$ and we use the formulas for implicit differentiation

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \quad .$$

In this problem

$$F_x = \cos(x + 2y + 3z) - 5yz$$

$$F_y = 2\cos(x + 2y + 3z) - 5xz$$

$$F_z = 3\cos(x + 2y + 3z) - 5xy$$

$$\frac{\partial z}{\partial x} = -\frac{\cos(x + 2y + 3z) - 5yz}{3\cos(x + 2y + 3z) - 5xy} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{2\cos(x + 2y + 3z) - 5xz}{3\cos(x + 2y + 3z) - 5xy} \quad .$$

These are the **answers**.

6. (13 points) Use polar coordinates to compute the double integral

$$\int\int_D xy\,dA \quad ,$$

where

$$D = \{(x,y)\,|\,x^2+y^2 \le 4\,,\, x \ge 0\,,\, y \ge 0\} \quad .$$

(Hint: recall the trig identity $\sin 2\theta = 2\sin\theta\cos\theta$)

Solution: Our region, in polar, is

$$D = \{(r,\theta)\,|\,0 \le r \le 2\,,\, 0 \le \theta \le \pi/2\}$$

So the integral becom

$$\int_0^{\pi/2}\int_0^2 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= (\int_0^2 r^3\,dr)(\int_0^{\pi/2} \cos\theta\sin\theta)\,d\theta)$$

Using the hint, $\cos\theta\sin\theta = (1/2)sin(2\theta)$, so we have

$$= (\int_0^2 r^3\,dr)(1/2)(\int_0^{\pi/2} \sin(2\theta)\,d\theta)$$

$$= (\frac{r^4}{4}\Big|_0^2)(\frac{-\cos 2\theta}{4}\Big|_0^{\pi/2}) = 2 \quad .$$

Ans.: 2.

7. (13 points) Use the transformation

$$x = 2u + v \quad , \quad y = u + 2v \quad ,$$

to evaluate the integral

$$\int\int_R (2x - y)\, dA$$

where R is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$.

Solution: The Jacobian is 3. The new triangle has vertices $(0,0)$, $(1,0)$ and $(0,1)$, so the region, in the uv-plane, is

$$R' = \{(u,v) \,|\, 0 \leq u \leq 1 \,,\, 0 \leq v \leq 1 - u\,\}$$

Using algebra, the **integrand** becomes $2x - y = 2(2u+v) - (u+2v) = 4u+2v-u-2v = 3u$ and the transformed integral is

$$\int_0^1 \int_0^{1-u} (3u)(3)\, dv\, du = 9 \int_0^1 \int_0^{1-u} u\, dv\, du = 9 \int_0^1 u(1-u)\, du = 9 \int_0^1 (u-u^2)\, du = \frac{3}{2} \quad .$$

Ans.: $\frac{3}{2}$.

8. (13 points) By using Stokes' theorem, or otherwise, evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad ,$$

where $\mathbf{F}$ is the vector field

$$\mathbf{F}(x, y, z) = \langle\, 2xy^2z^2 \,,\, 2x^2yz^2 \,,\, 2x^2y^2z \,\rangle \quad ,$$

and C is the **closed** curve going from $(1, 0, 1)$ to $(3, 4, 9)$, and then from $(3, 4, 9)$ to $(-1, 4, 11)$, and then from $(-1, 4, 11)$ to $(5, 2, 11)$ and finally from $(5, 2, 11)$ back to the starting point $(1, 0, 1)$. Explain everything!

Solution: $curl\,\mathbf{F} = \langle 0, 0, 0\rangle$ (you do it!). By Stokes' Theorem $\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S curl\,\mathbf{F} \cdot d\mathbf{S}$, but if the integrand is 0 the integral is 0.

Ans.: 0.

9.

(a) (4 points) Compute the surface integral

$$\int\int_S 8\,dS \quad ,$$

where S is the sphere $(x-1)^2+(y+4)^2+(z-9)^2=100$.

(b) (4 points) Compute the triple integral

$$\int\int\int_E 30\,dV \quad ,$$

where E is the ball $\{\,(x,y,z)\,|\,(x-1)^2+(y+4)^2+(z-9)^2\leq 100\,\}$.

(c) (4 points) Compute the line integral

$$\int_C 3\,ds \quad ,$$

where C is the circumference of the region $\{(x,y)\,|\,x^2+y^2\leq 4\,,\,y\geq 0\}$.

Sol.to (a): This is 8 times the surface area of the given sphere that has radius 10 so it equals $8(4\pi(10)^2)=3200\pi$.

Sol.to (b): This is 30 times the volume of the given sphere, so it equals $30(4/3)\pi(10)^3=40000\pi$.

Sol.to (c): This is 3 times the circumference of the the **semi-disc** of radius 2, that consists of the semi-circle of radius 2 and the base that is its diameter, the line segment between $(-2,0)$ and $(2,0)$. So it equals $3(2\pi+2(2))=12+6\pi$.

10. (12 points) Evaluate the iterated integral by converting it to polar coordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx \quad .$$

Solution: The region, in polar is,

$$D = \{(r,\theta)\,|\, 0 \le \theta \le \pi/2\,,\ 0 \le r \le 1\,\}$$

So the integral becom

$$\int_0^{\pi/2} \int_0^1 e^{r^2}\, r\, dr\, d\theta = (\int_0^1 re^{r^2}\, dr)(\int_0^{\pi/2} d\theta)$$

$$= \left((1/2)e^{r^2} \Big|_0^1 \right) (\pi/2) = \frac{\pi}{4}(e-1) \quad .$$

Ans.: $\frac{\pi}{4}(e-1)$.

11. (12 points) Evaluate the surface integral

$$\int\int_S \sqrt{3}\, x \, dS \quad ,$$

where S is the triangular region with vertices $(1,0,0),(0,1,0),(0,0,1)$.

Solution: The surface is the plane passing through the three given points. It is $x+y+z=1$ (you do it!), so the surface explicitly is $z = 1-x-y$. The projection of the triangle on the xy-plane is the triangle whose vertices are $(0,0),(1,0),(0,1)$. So the floor-region (in type-I style) is

$$\{(x,y)\,|\,0 \leq x \leq 1\,,\,0 \leq y \leq 1-x\,\} \quad .$$

The formula for a surface integral for a surface given explicitly ($z = f(x,y)$) is:

$$\int\int_D F(x,y,z)\sqrt{1+(z_x)^2+(z_y)^2}\, dA \quad ,$$

where D is the floor region, and we plug-in for z, $f(x,y)$. Here there is no z, so the integral is

$$\int\int_D \sqrt{3}x\sqrt{1+(-1)^2+(-1)^2}\, dA = 3\int_0^1\int_0^{1-x} x\, dy\, dx = 3\int_0^1 x(1-x)dx = 3\int_0^1 (x-x^2)dx = \frac{1}{2} \quad .$$

Ans.: $\frac{1}{2}$.

12. (12 points) Determine whether or not the vector field

$$F(x,y,z) = (e^x + yz)\,\mathbf{i} + (e^y + xz)\,\mathbf{j} + (e^z + xy)\,\mathbf{k}$$

is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

Solution: $curl\mathbf{F} = \langle 0,0,0\rangle$ (you do it!), hence the vector field $\mathbf{F}$ is conservative.

To find the potential function f, we need to find a function f such that

$$f_x = e^x + yz \quad , \quad f_y = e^y + xz \quad , \quad f_z = e^z + xy \quad ,$$

From $f_x = e^x + yz$, we get $f = \int (e^x + yz)\,dx = e^x + xyz + g(y,z)$, where $g(y,z)$ is yet to be determined. From $f_y = e^y + xz$ we get $xz + g_y = e^y + xz$. Using algebra, we get: $g_y = e^y$, so $g = \int e^y\,dy = e^y + h(z)$, where $h(z)$ is yet to be determined. So, now we have $f = e^x + e^y + xyz + h(z)$.

Using $f_z = e^z + xy$ we get $xy + h'(z) = e^z + xy$, so $h'(z) = e^z$ and hence $h(z) = e^z + C$, but we get make $C = 0$, and we get that a potential function is

$$f = e^x + e^y + e^z + xyz \quad .$$

Ans.: The vector field $\mathbf{F}$ is conservative, since $curl\,\mathbf{F} = \mathbf{0}$, nd the potential function is $f = e^x + e^y + e^z + xyz$.

13. (12 points) By using Green's theorem, or otherwise, evaluate the line integral

$$\int_C e^y\,dx + 2xe^y\,dy \quad ,$$

where C goes from $(0,0)$ to $(1,0)$, then from $(1,0)$ to $(1,1)$, then from $(1,1)$ to $(0,1)$, and then from $(0,1)$ back to $(0,0)$.

Solution: Recall that Green's Theorem says that

$$\int_C P\,dx + Q\,dy = \int\int_R (Q_x - P_y)\,dA$$

where R is the region inside C and C is a closed curve traveled in the counterclockwise direction.

$$\int_C e^y\,dx + 2xe^y\,dy = \int\int_R (2e^y - e^y)\,dA = \int\int_R e^y\,dA$$

R is the rectangle

$$R = \{(x,y)\,|\,0 \le x \le 1\,,\,0 \le y \le 1\} \quad ,$$

so our double-integral becomes the iterated integral

$$\int_0^1 \int_0^1 e^y\,dy\,dx = \int_0^1 (e-1)dx = e-1 \quad .$$

Ans.: $e-1$.

14. (12 points) Show that the line integral

$$\int_C 2x \sin y \, dx \; + \; (x^2 \cos y - 3y^2) \, dy \quad ,$$

is independent of the path C, and evaluate it if C is *any* path from $(1,0)$ to $(0,2)$.

Solution:

Here $P = 2x \sin y$, $Q = x^2 \cos y - 3y^2$. $P_y = 2x \cos y$, $Q_x = 2x \cos y$. Since $Q_x = P_y$, the vector-field $\langle P, Q \rangle$ is conservative, and hence it makes sense to look for a potential function $f(x,y)$ such that $\bigtriangledown f = \langle P, Q \rangle$, in other words $f_x = 2x \sin y$, and $f_y = x^2 \cos y - 3y^2$.

From $f_x = 2x \sin y$ we get $f = x^2 \sin y + g(y)$. From $f_y = x^2 \cos y - 3y^2$ we get $x^2 \cos y + g'(y) = x^2 \cos y - 3y^2$ so $g'(y) = -3y^2$ and so $g(y) = -y^3$. Going back to $f = x^2 \sin y + g(y)$ we get that the potential function is

$$f(x,y) = x^2 \sin y - y^3 \quad .$$

By the **Fundamental Theorem for Line-Integrals** the value of the integral is $f(end) - f(start)$, so

$$f(0,2) - f(1,0) = -8 - 0 = -8 \quad .$$

Ans.: -8.

15. (12 points) Evaluate

$$\int\int\int_B 5\,(x^2+y^2+z^2)^2\,dV$$

where B is the ball

$$\{\,(x,y,z)\,|\,x^2+y^2+z^2 \le 4\,\} \quad .$$

Solution: Converting to spherical coordinates we have

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{2} 5(\rho^2)^2\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

This equals

$$5\int_0^{\pi}\int_0^{2\pi}\int_0^{2} \rho^6 \sin\phi\,d\rho\,d\theta\,d\phi \quad .$$

By the separation trick this is

$$\left(5\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{2}\rho^6\,d\rho\right)\left(\int_0^{2\pi}d\theta\right) = \frac{2560\pi}{7} \quad .$$

Ans.: $\frac{2560\pi}{7}$.

16. (12 points) Find the local maximum and minimum point(s), the local maximum and minimum values, and saddle point(s) of the function

$$f(x, y) = 6y^2 - 2y^3 + 3x^2 + 6xy \quad .$$

Solution:

$$f_x = 6x + 6y \quad , \quad f_y = 12y - 6y^2 + 6x \quad .$$

. For future reference:

$$f_{xx} = 6 \quad , \quad f_{xy} = 6 \quad , \quad f_{yy} = 12 - 12y \quad .$$

We have to solve the system

$$6x + 6y = 0 \quad , \quad 12y - 6y^2 + 6x = 0 \quad .$$

From the first equation we have $y = -x$. Plugging this into the second equation, we get $-12x - 6x^2 + 6x = 0$, so $-6x^2 - 6x = 0$ so $x^2 + x = 0$ which is $x(x+1) = 0$ that has two solutions:

$$x = 0 \quad y = -0 = 0 \quad ;$$

$$x = -1 \quad y = -(-1) = 1 \quad .$$

So there are two critical points: $(0, 0)$ and $(-1, 1)$.

At $(0, 0)$, $f_{xx} = 6, f_{xy} = 6, f_{yy} = 12$, and so $D = 6 \cdot 12 - 6^2 = 36 > 0$ and since $f_{xx} > 0$, this is a **local minimum point**. The value there is $f(0, 0) = 0$.

At $(-1, 1)$, $f_{xx} = 6, f_{xy} = 6, f_{yy} = 0$, and so $D = 6 \cdot 0 - 6^2 = -36 < 0$ and hence this is a **saddle point**.

Ans.:

Local Maximum Points: none ; Local Maximum Values: none.

Local Minimum Point: $(0, 0)$; Local Maximum Value: 0.

Saddle points: $(-1, 1)$.