Complete SOLUTIONS to:

MATH 251 (1-3), Dr. Z. , Mid-Term #2, 10:20-11:40 , Mon., Nov. 20, 2006

1. Determine whether or not

$$\mathbf{F} = (e^x y - \cos(x+y) + 1)\mathbf{i} + (e^x - \cos(x+y) + 1)\mathbf{j}$$

is a conservative vector field. If it is, find a function f that $\mathbf{F} = \bigtriangledown f$.

Ans.: F is: **conservative**.

(If applicable): $f = e^x y - sin(x+y) + x + y$

Solution to First Part of **1**.

Recall that the condition for $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ (or $F = \langle P, Q \rangle$) to be **conservative** is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad .$$

In this problem,

$$P = e^x y - \cos(x+y) + 1 \quad , \quad Q = e^x - \cos(x+y) + 1 \quad ,$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[e^x y - \cos(x+y) + 1] = e^x + \sin(x+y) \quad ,$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[e^x - \cos(x+y) + 1] = e^x + \sin(x+y) \quad ,$$

Since these are the *same*, **F** is indeed **conservative**.

Solution to Second Part of **1**. We have to find the (so-called **potential function**) $f(x,y)$ such that $\bigtriangledown f = \mathbf{F}$. That means $\langle\, f_x \,,\, f_y \,\rangle = \langle\, P, Q \,\rangle$. Spelling-out,

$$f_x = P \quad , \quad f_y = Q \quad .$$

Let's first try to make the **first** equation happy. The derivative of f with respect to the variable x is P. So f is the **anti-derivative** of P w.r.t. to x:

$$f = \int (e^x y - \cos(x+y) + 1)\, dx = e^x y - \sin(x+y) + x + g(y) \quad ,$$

(we used the fact that $\int \cos(x+a)\, dx = \sin(x+a) + C$ and $\int 1\, dx = x + C$. The $g(y)$ plays the role of the "arbitrary constant", since this was an integration with respect to x, so g is constant from the point of view of the variable x, but of course usually depends on y.

So we can say that we "almost" know what f is explicitly:

$$f = e^x y - \sin(x+y) + x + g(y) \quad . \qquad (TentativeFormForf(x,y))$$

All we need is to figure out what $g(y)$ is. Once we will know that, we would have to **backtrack** into the above equation.

In order to find what $g(y)$ is we need to make the **second equation**:

$$f_y = Q \quad ,$$

come true. Using the above *tentative form* of $f(x,y)$, we get that f_y equals

$$f_y = e^x - \cos(x+y) + g'(y) \quad .$$

Setting this equal to Q, yields

$$e^x - \cos(x+y) + g'(y) = e^x - \cos(x+y) + 1 \quad .$$

Using algebra, we get $g'(y) = 1$. To get $g(y)$ we find the **anti-derivative** (alias indefinite integral) w.r.t. to y to get $g(y) = \int 1 \cdot dy = y + C$ (but you really don't care about the C, so you can make it 0), so $g(y) = y$. Now **going back** to $TentativeFormForf(x,y)$, and **implementing** the newly found $g(y)$, we get

$$f = e^x y - \sin(x+y) + x + y \quad , \qquad (FinalFormForf(x,y))$$

Ans. to Second Part of (1): $f = e^x y - \sin(x+y) + x + y$.

2. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad ,$$

where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k} \quad ,$$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k} \quad , \quad 0 \leq t \leq 1 \quad .$$

Ans.: 3. [type number].

Solution to 2.: This is a **line-integral** of the **vector-field** kind, where the **inputs** are a vector-field and a *curve* and the output is a **number**. It can be written in long-hand as

$$\int_C P\, dx + Q\, dy + R\, dz$$

(where $F = \langle P, Q, R \rangle$). Please do not confuse with the other kind of line-integral whose inputs are a **function** $f(x, y, z)$ and a curve and looks like

$$\int_C f(x, y, z)\, ds \quad ,$$

featuring the **arc-length** differential ds which is $ds = |\mathbf{r}'(t)|\, dt$. In **this problem** we do not need $|\mathbf{r}'(t)|$.

Going back to our problem

$$P = y \quad , \quad Q = x \quad , \quad R = z \quad ;$$

$$x = t \quad , \quad y = t^2 \quad , \quad z = 2t \quad ;$$

$$dx = dt \quad , \quad dy = (2t)\, dt \quad , \quad dz = 2\, dt \quad .$$

Plugging-in everything

$$\int_C P\, dx + Q\, dy + R\, dz = \int_0^1 (t^2)\, dt + (t)(2t)\, dt + (2t)(2)\, dt = \int_0^1 [t^2 + 2t^2 + 4t]\, dt$$

$$= \int_0^1 [3t^2 + 4t]\, dt = t^3 + 2t^2 \Big|_0^1 = 1^3 + 2(1)^2 - 0 = 1 + 2 = 3.$$

Ans. to 2.: 3 [type number].

Comments: Some clever people found a **short-cut**. They took a chance that **F** was conservative (by now you know how to decide it beforehand by taking *curl* **F** and seeing whether it is the **0** vector). Anyway, in this (lucky!, it does not always happen, in which case you must do it the direct way as above) case, you can find that the potential function f is $f(x, y, z) = xy + z^2/2$. The start of our curve (plug-in $t = 0$ into $\mathbf{r}$) is the point $(0, 0, 0)$. The end of our curve (plug-in $t = 1$ into $\mathbf{r}$) is $(1, 1, 2)$, so using the **Fundamental Theorem of Line-Integrals**, we get:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 2) - f(0, 0, 0) = 1 \cdot 1 + 2^2/2 - 0 = 3 \quad .$$

The second way is perfectly correct, and the few (clever) people who did it (correctly) got full credit.

Common mistakes:

1. Confuse with the other kind of line integrals and computing $|\mathbf{r}'(t)|$ which is completely irrelevant.

2. There is lots of plugging-in to do in this problem, some people were sloppy and plugged-in wrong things.

3. Evaluate

$$\int\int\int_E 7(x^2+y^2)^2\,dV \quad ,$$

where E is the solid that lies within the cylinder $x^2+y^2=4$, above the plane $z=0$, and below the cone $z^2=9x^2+9y^2$.

Ans.: 768π [type number].

Solution to 3. This calls for **cylindrical** coordinates. All we need from the dictionary is that the 'phrase' x^2+y^2 equals r^2 and that $dV = r\,dz\,dr\,d\theta$.

The "floor-plan" is the inside of the circle $x^2+y^2=2^2$. which in cylindrical (or polar) is $r=2$ and so θ takes the **default** range $0 \le \theta \le 2\pi$. Our solid goes from the floor up to $z^2=9x^2+9y^2$ which is $z^2=9r^2$ i.e. $z=\pm 3r$ but everything is **above** the plane $z=0$ so we discard the **minus**-option and get that the range in z is $0 \le z \le 3$. So the description of the **region of volume-integration** in **cylindrical** coordinates is:

$$E = \{(r,\theta,z)\,|\,0 \le \theta \le 2\pi\,,\,0 \le r \le 2\,,\,0 \le z \le 3r\,\} \quad .$$

And our triple-integral becomes the **iterated-integral**

$$\int_0^{2\pi}\int_0^2\int_0^{3r} 7(r^2)^2 r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^{3r} 7r^5\,dz\,dr\,d\theta \quad .$$

The **inside-integral** is:

$$\int_0^{3r} 7r^5\,dz = 7r^5\int_0^{3r} dz = 7r^5\left([z]\Big|_0^{3r}\right) = 7r^5\cdot(3r) = 21r^6 \quad .$$

The **middle-integral** is:

$$\int_0^2 [AnswerToInnerIntegral]\,dr \quad .$$

$$\int_0^2 [21r^6]\,dr = 21\frac{r^7}{7}\Big|_0^2 = 3r^7\Big|_0^2 = 3(2)^7 - 3(0)^7 = 3\cdot 2^7 = 384.$$

The **outside-integral** is

$$= \int_0^{2\pi}[AnswerToMiddleIntegral]d\theta = \int_0^{2\pi}[384]d\theta = 384\int_0^{2\pi} d\theta = 384[\theta]\Big|_0^{2\pi} = 384(2\pi - 0) = 768\pi.$$

Ans. to 3.: 768π [type number].

4. Evaluate the iterated integral

$$\int_0^1 \int_x^{2x} \int_0^{x+y} (6x+6y)\, dz\, dy\, dx \quad .$$

Ans.: $\frac{19}{2}$ [type number].

Solution to 4.: First we do the **inner-integral**

$$\int_0^{x+y} (6x+6y)\, dz = 6(x+y) \int_0^{x+y} 1\, dz = 6(x+y) \left(z \Big|_0^{x+y} \right) = 6(x+y)^2 \quad .$$

Now we do the **middle-integral**:

$$\int_x^{2x} [AnswerToInnerIntegral] dy = \int_x^{2x} [6(x+y)^2] dy = 6 \frac{(x+y)^3}{3} \Big|_x^{2x} =$$

$$2(x+y)^3 \Big|_x^{2x} = 2(x+2x)^3 - 2(x+x)^3 = 2(3x)^3 - 2(2x)^3 = 2(27x^3 - 8x^3) = 2(27-8)x^3 = 38x^3 \quad .$$

Note: Here we used the "canned-formula"

$$\int (ay+b)^n\, dy = \frac{(ay+b)^{n+1}}{(n+1)a} \quad .$$

This is much faster than expanding $(x+y)^2$ into $x^2+2xy+y^2$ and doing it one-piece-at-a-time. Even the best-performing students (who got perfect score) did it the "long way").

Finally the answer to the **outside integral** (which is the final answer) is:

$$\int_0^1 [AnswerToMiddleIntegral] dx = \int_0^1 [38x^3] dx = 38 \frac{x^4}{4} \Big|_0^1 = \frac{19}{2} x^4 \Big|_0^1 =$$

$$\frac{19}{2}(1^4 - 0^4) = \frac{19}{2} \quad .$$

Ans.: $\frac{19}{2}$ [type number].

Common Mistakes: If you do it the long way the algebra is messy and it is easy to mess-up somewhere in the calculations.

5. Use the given transformation to evaluate the integral

$$\int\int_R 4(2x+y)^2\,dA \quad ,$$

where R is the triangular region with vertices $(0,0)$,$(2,-3)$, $(3,-5)$; $x = 3u - v$, $y = -5u + 2v$.

Ans.: 1 [type number].

Solution to 5.: The **Jacobian is**:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -5 & 2 \end{vmatrix} = (3)(2) - (-1)(-5) = 1 \quad .$$

We now have to find the **counterparts** of the three vertices in the (u, v)-plane.

For the point $(0, 0)$, solving

$$3u - v = 0 \quad , \quad -5u + 2v = 0 \quad ,$$

we get $u = 0, v = 0$ (you do it!), so $(0, 0)$ corresponds to $(0, 0)$.

For the point $(2, -3)$, solving

$$3u - v = 2 \quad , \quad -5u + 2v = -3 \quad ,$$

we get $u = 1, v = 1$ (you do it!), so $(2, -3)$ corresponds to $(1, 1)$.

For the point $(3, -5)$, solving

$$3u - v = 3 \quad , \quad -5u + 2v = -5 \quad ,$$

we get $u = 1, v = 0$ (you do it!), so $(3, -5)$ corresponds to $(1, 0)$.

Now we have a much simpler triangle!

(Note: the point is that the new vertices, and triangle, are much simpler than the original, if you get complicated points (because you messed-up the algebra!) it means that something is **fishy** and you should check your work.)

We also have to convert the **integrand** from the (x, y)-language to the (u, v)-language:

$$4(2x+y)^2 = 4(2[3u-v] + [-5u+2v])^2 = 4(6u - 2v - 5u + 2v)^2 = 4u^2 \quad .$$

So using **multi-variable change-of-variables** we get

$$\int\int_R 4(2x+y)^2\,dA \quad =$$

$$= \int\int_{R'} J \cdot 4u^2 \, dA =$$

$$= \int\int_{R'} 1 \cdot 4u^2 \, dA = \int\int_{R'} 4u^2 \, dA \quad .$$

where R' is the **simplified triangle** in the (u, v)-plane whose vertices are $(0, 0), (1, 0), (1, 1)$. Drawing it (do it!), it is simpler to express it as a **type I** region:

$$R' = \{(u, v) \,|\, 0 \le u \le 1 \,,\, 0 \le v \le u\} \quad .$$

Our (simplified) double-integral becomes the **iterated integral**

$$\int_0^1 \int_0^u 4u^2 \, dv \, du \quad .$$

The **inner-integral** is:

$$\int_0^u 4u^2 \, dv = 4u^2 \int_0^u dv = 4u^2 [v\Big|_0^u] = 4u^2[u - 0] = 4u^3 \quad .$$

The **outside-integral** (and the final answer) is:

$$\int_0^1 [AnswerToInnerIntegral] \, du = \int_0^1 [4u^3] \, du = u^4\Big|_0^1 = 1^4 - 0^4 = 1 \quad .$$

Ans.: 1 [type number].

Some Mistakes People Made

1. Mess-up the (easy!) computation of the Jacobian $(3)(2) - (-1)(-5)$ and thinking that is is 11 (by miscounting the number of minus-signs).

2. Forget about the Jacobian altogether. In this problem they "lucked-out" since the Jacobian happens (by chance) to be 1 so they got the right answer, but I still took points off, since usually this does not happen.

3. Mess-up the algebra and get complicated (and wrong, of course) regions and waste lots of time in doing a much too complicated problem. Please use your **common sense**, if it is getting too complicated, you are probably on the wrong track.

4. Trying to do it directly staying in the (x, y)-plane. They got no (or little) credit, since you were asked to do it with the suggested method. Besides none of them succeeded since doing it directly is much more complicated!

6. Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy^2 \, dy \, dx$$

Ans.: $\frac{2}{15}$ [type number].

Solution to 6.: The region of integration, in the usual **rectangular** coordinates is:

$$E = \{(x,y) \,|\, 0 \le x \le 1 \,,\, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\} \quad .$$

If you draw it (do it!), you will see that it is the inside of a **semi-circle**, center origin, radius 1, that lies in the **right half-plane** (since x goes from 0 to 1, hence is always **positive**). This region, in polar coordinates is expressed as

$$E = \{(r,\theta) \,|\, -\pi/2 \le \theta \le \pi/2 \,,\, 0 \le r \le 1\} \quad .$$

Converting our integral to **polar**, using the **dictionary**

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad dA = r\, dr\, d\theta \quad ,$$

we get that our integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 r^4 \sin^2\theta \cos\theta dr \, d\theta \quad .$$

The **inner-integral** is:

$$\sin^2\theta\cos\theta \int_0^1 r^4 dr = \sin^2\theta\cos\theta \frac{r^5}{5}\Big|_0^1 =$$

$$\frac{1}{5}\sin^2\theta\cos\theta \quad .$$

The **outer-integral** (and the final answer is):

$$\int_{-\pi/2}^{\pi/2} [AnswerToInnerIntegral] \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{5}\sin^2\theta\cos\theta \, d\theta \quad .$$

Using the **substitution** $u = \sin\theta$ (so $\cos\theta\, d\theta = du$ and the limits are $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$) this equals

$$\frac{1}{5}\int_{-1}^1 u^2 \, du = \frac{1}{5}\frac{u^3}{3}\Big|_{-1}^1 = \frac{1}{15}(1^3 - (-1)^3) = \frac{2}{15} \quad .$$

Ans.: $\frac{2}{15}$ [type number].

7. Evaluate the iterated integral

$$\int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy \quad .$$

(Hint: Not even Dr. Z. can do $\int e^{x^2}\, dx$, so you must be clever!)

Ans.: $e^4 - 1$ [type number].

Solution to 7.: The region of integration, let's call D is

$$D = \{(x,y)\,|\, 0 \le y \le 4 \quad , \quad y/2 \le x \le 2\}$$

This is expressed in type II-style. If you draw it (do it!), you would get a triangle whose vertices are $(0,0)$, $(2,0)$, and $(2,4)$. Note that the hypotenuse is the line $y = 2x$. Expressing D as a **type I** region, we get

$$D = \{(x,y)\,|\, 0 \le x \le 2 \quad , \quad 0 \le y \le 2x\} \quad .$$

So our integral, written in type I-style is the **iterated integral**

$$\int_0^2 \int_0^{2x} e^{x^2}\, dy\, dx \quad .$$

The inner-integral is:

$$\int_0^{2x} e^{x^2}\, dy = e^{x^2} \int_0^{2x} dy = e^{x^2} \left(y \Big|_0^{2x} \right) = e^{x^2}(2x - 0) = 2x e^{x^2} \quad .$$

The **outer-integral** is;

$$\int_0^2 [AnswerToInnerIntegral]\, dx = \int_0^2 e^{x^2}\, (2x dx) \quad .$$

This is done with the **substitution** $u = x^2$ so $du = 2x\, dx$, $x = 0$ becomes $u = 0^2 = 0$ and $x = 2$ becomes $u = 2^2 = 4$. Our integral in the u-language is:

$$\int_0^4 e^u\, du = e^u \Big|_1^4 = e^4 - e^0 = e^4 - 1 \quad .$$

Ans: $e^4 - 1$ [type number].

Some Mistakes People Made: 1. Try to do it directly (hopeless!). 2. Draw the wrong picture and mess-up the integration limits. 3. Unable to do the integral $\int 2xe^{x^2}$ (some people tried integration-by-parts, which is **not** the right way for this integral).

8. Calculate the double integral

$$\int\int_R \frac{3xy^2}{x^2+1}\, dA \quad ,$$

$$R = \{(x,y)\,|\, 0 \le x \le 1\,,\, -1 \le y \le 1\,\} \quad .$$

Ans.: $\ln 2$ [type number].

Solution to 8.: Converting the given double-integral into an **iterated integral** we get:

$$\int_0^1 \int_{-1}^1 \frac{3xy^2}{x^2+1}\, dydx \quad .$$

The **inner integral** is

$$\int_{-1}^1 \frac{3xy^2}{x^2+1}\, dy = \frac{3x}{x^2+1} \int_{-1}^1 y^2\, dy$$

$$= \frac{3x}{x^2+1} \frac{y^3}{3} \Big|_{-1}^1 = \frac{x}{x^2+1}(1^3 - (-1)^3) = \frac{2x}{x^2+1} \quad .$$

The **outer integral** is

$$\int_0^1 [AnswerToInnerIntegral]dx = \int_0^1 \frac{2x}{x^2+1} dx \quad .$$

This is done with the **substitution** $u = x^2+1$ so $du = 2x\,dx$, $x=0$ becomes $u=1$, $x=1$ becomes $u=2$ and we have

$$\int_0^1 \frac{2x\,dx}{x^2+1} = \int_1^2 \frac{1}{u} du = \ln|u| \Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

Ans.: $\ln 2$ [type number].

Short Cut: This integral falls under the type of

$$\int \frac{dCabin}{Cabin} = \log Cabin \quad .$$

[In advanced math log means ln].

$$\int_0^1 \frac{2x}{x^2+1} dx = \int_0^1 \frac{d(x^2+1)}{x^2+1} = \ln|x^2+1| \Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$$

Some Mistakes People Made: 1. Forgot how to do the simple integration $\int \frac{2x}{x^2+1}\, dx$. Some tried integration-by-parts and some used their own **illegal** "product-rule": e.g.

$$\int_0^1 \frac{2x}{x^2+1} dx = \left(\int_0^1 (2x) dx\right) \left(\int_0^1 \frac{1}{x^2+1} dx\right) \quad .$$

Remember: if you are stuck, admit it, and don't lose important points (or have the chance of killing people) by "winging it".

9. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = xy^2$ subject to the constraint $2x^2 + y^2 = 6$.

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maximum value: 4 [type number].

minimum value: -4 [type number].

Solution to 9.: $f = xy^2$, $g = 2x^2 + y^2$. Here $\bigtriangledown f = \langle y^2\,,\,2xy\rangle$ $\bigtriangledown g = \langle 4x\,,\,2y\rangle$. We need to solve $\bigtriangledown f = \lambda \bigtriangledown g$ plus the **constraint equation** $2x^2 + y^2 = 6$. This means that we have to solve the **system**

$$y^2 = 4\lambda x \quad , \quad 2xy = 2\lambda y \quad , \quad 2x^2 + y^2 = 6 \quad .$$

It is a good idea to exploit the first two equations first. Dividing the first equation by the second, we get:

$$\frac{y^2}{2xy} = \frac{4\lambda x}{2\lambda y} \quad .$$

(When we divide , we have to explore separately the possibility that the thing we divide by is 0, in this case $2xy = 0$ means $x = 0$ or $y = 0$. $x = 0$ does not yield anything, but $y = 0$ yields $x = \pm\sqrt{3}$, so in addition to the solutions that are coming up, we have to include $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$ below).

Simplifying

$$\frac{y}{2x} = \frac{2x}{y} \quad .$$

(Note: of course when we divide we must not divide by zero so we have to rule out the case that x and/or y are zero, but it is easy to see that it can never happen because of the last equation).

Cross-multiplying:

$$y^2 - (2x)^2 = 0 \quad .$$

factoring (using the famous $a^2 - b^2 = (a-b)(a+b)$):

$$(y - 2x)(y + 2x) = 0 \quad .$$

Which means that either $y = 2x$ or $y = -2x$. Plugging these into the **last** equation we get

$$2x^2 + (\pm 2x)^2 = 6 \quad ,$$

so $2x^2 + 4x^2 = 6$ which is $6x^2 = 6$, and dividing both sides by 6 we get

$$x^2 = 1 \quad ,$$

which has **two** solutions $x = -1$ and $x = 1$. Plugging back into $y = 2x$ and $y = -2x$ we get four **finalists**:

$$(-1, -2) \quad , \quad (-1, 2) \quad , (1, -2) \quad , \quad (1, 2) \quad .$$

Plugging-into $f(x, y)$ we do the **final contest**:

$$f(-1,-2) = (-1)(-2)^2 = -4\,,\; f(-1,2) = (-1)(2)^2 = -4\,,\; f(1,-2) = (1)(-2)^2 = 4\,,\; f(1,2) = (1)(2)^2 = 4 \quad ,$$

$$f(-\sqrt{3},0) = 0 \quad , f(\sqrt{3},0) = 0 \quad .$$

Amongst these the **maximum value** is 4 and the **minimum value** is -4.

Ans.: max. value is 4 [type number] min. value is -4 [type number].

Mistakes People make: Trouble with the algebra. Remember the first thing to try is to **divide** the first equation by the second (to get rid of λ).

10. Find the local maximum and minimum **values**, and saddle point(s) of the function $f(x, y) = x^4 + y^4 - 4xy + 2$.

Local maximum value(s): none

Local minimum value(s): 0 [type number]

saddle point(s): $(0, 0)$ [type point]

Solution of 10.: Here $f = x^4 + y^4 - 4xy + 2$ so

$$f_x = 4x^3 - 4y \quad , \quad f_y = 4y^3 - 4x \quad .$$

We also need, for later,

$$f_{xx} = 12x^2 \quad , \quad f_{xy} = -4 \quad , \quad f_{yy} = 12y^2 \quad .$$

We first have to solve the **system** of two equations and two unknowns, $f_x = 0$, $f_y = 0$, which in this problem is:

$$4x^3 - 4y = 0 \quad , \quad 4y^3 - 4x = 0 \quad .$$

Simplifying:

$$x^3 - y = 0 \quad , \quad y^3 - x = 0 \quad .$$

So from the first equation, $y = x^3$ and plugging into the second $(x^3)^3 - x = 0$ so $x^9 - x = 0$. Factoring gives $x(x^8 - 1) = 0$. Recall that $x^{even} = 1$ has **two** roots $x = -1$ and $x = 1$, (on the other hand $x^{odd} = 1$ only has **one** solution $x = 1$. So we have **three** solutions altogether:

$$x = -1 \quad , \quad x = 0 \quad , \quad x = 1 \quad .$$

.

But since $y = x^3$ **always**:

$x = -1$ implies $y = (-1)^3 = -1$ yielding the point: $(-1, -1)$.

$x = 0$ implies $y = (0)^3 = 0$ yielding the point: $(0, 0)$,

$x = 1$, implies $y = (1)^3 = 1$ yielding the point: $(1, 1)$.

Now it is time to investigate each of our points one at a time.

Recall the **discriminant**

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 \quad .$$

For the point $(-1,-1)$ $f_{xx} = 12(-1)^2 = 12 > 0$ $f_{yy} = 12(-1)^2 = 12$, $f_{xy} = -4$. So $D(-1,-1) = (12)(12) - (-4)^2 = 144 - 16 = 128 > 0$, and since $f_{xx}(-1,-1) > 0$ we conclude that this point is a **local minimum** and the corresponding **value** is $f(-1,-1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 2 = 0$.

For the point $(0,0)$ $f_{xx} = 12(0)^2 = 0$ $f_{yy} = 12(0)^2 = 0$, $f_{xy} = -4$. So $D(0,0) = (0)(0) - (-4)^2 = 0 - 16 = -16 < 0$, we conclude that this point is a **saddle point**.

For the point $(1,1)$ $f_{xx} = 12(1)^2 = 12 > 0$ $f_{yy} = 12(1)^2 = 12$, $f_{xy} = -4$. So $D(1,1) = (12)(12) - (-4)^2 = 144 - 16 = 128 > 0$, and since $f_{xx}(1,1) > 0$ we conclude that this point is a **local minimum** and the corresponding **value** is $f(1,1) = (1)^4 + (1)^4 - 4(1)(1) + 2 = 0$.

Ans.: There are no local maxima. The local minimum value is 0 (that takes place at $(-1,-1)$ and at $(1,1)$) and there is one saddle point at $(0,0)$.

Some Mistakes People Make: When doing the algebra a few people forgot the $x = 0$ option. Also When plugging-in to get the **values**, plug-into $f(x,y)$ not $D(x,y)$. The actual values of D are not important, only whether they are positive or negative.