Complete Solutions to

MATH 251 (4-6), Dr. Z. , Mid-Term #2, 12:00-1:20 , Mon., Nov. 20, 2006

[Version of Dec. 3, 2007 [some typos corrected, please discard earlier version]

1. (a) Find a function f such that $\mathbf{F} = \bigtriangledown f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x,y,z) = 2xy^2z^2\,\mathbf{i} + 2x^2yz^2\,\mathbf{j} + 2x^2y^2z\,\mathbf{k} \quad ,$$

$$C : x = t^3 \quad , \quad y = t^2+1 \quad , \quad z = 2t+1 \quad , \quad 0 \le t \le 1 \quad .$$

Ans. to (a): $f = x^2y^2z^2$ [type function]

Ans. to (b): 36 [type number]

Solution to 1.(a): $P = 2xy^2z^2$, $Q = 2x^2yz^2$, $R = 2x^2y^2z$. We have to find a function $f(x,y,z)$ such that

$$f_x = P \quad , \quad f_y = Q \quad , \quad f_z = R \quad .$$

Using the first equation, we get

$$f = \int P\,dx = \int 2xy^2z^2\,dx = y^2z^2 \int 2x\,dx = y^2z^2x^2 + g(y,z) \quad ,$$

where $g(y,z)$ plays the role of the **arbitrary constant**, since it does **not** depend on x.

So the **tentative form** of f is

$$f = x^2y^2z^2 + g(y,z) \quad .$$

Once we would know $g(y,z)$ we would be all set.

Now, let's make the second equation happy: $f_y = Q$. Using the above tentative form of f:

$$f_y = 2x^2yz^2 + g_y(y,z) \quad ,$$

and setting this equal to Q we get

$$2x^2yz^2 + g_y(y,z) = 2x^2yz^2 \quad .$$

Using **algebra** we get:

$$g_y(y,z) = 0 \quad .$$

To get $g(y,z)$ we integrate w.r.t. y getting

$$g(y,z) = \int 0\,dy = 0 + h(z) \quad ,$$

where $h(z)$ plays the role of the arbitrary constant, since it does **not** depend on y. Going back to the above **tentative form** of f, and replacing $g(y,z)$ by what we just got, we get a **new**, less tentative form for f:

$$f = x^2y^2z^2 + h(z) \quad .$$

Now it is time to take advantage of the last equation: $f_z = R$. Using the recently obtained less tentative form of f, we have

$$f_z = 2x^2y^2z + h'(z) \quad .$$

Setting it equal to R, we get

$$2x^2y^2z + h'(z) = 2x^2y^2z \quad ,$$

from which we get $h'(z) = 0$. Integrating w.r.t. to z, we get $h(z) = \int 0\,dz = C$, but we can safely set $C = 0$, since we are only intersetd in **a** function not in **all** of them. Going back to the less tentative form of f, and incorporating our recent discovery that $h(z) = 0$, we get

$$f = x^2y^2z^2 + 0 = x^2y^2z^2 \quad .$$

Ans. to 1(a): $f(x,y,z) = x^2y^2z^2$.

Sol. to 1(b): Using the **Fundamental Theorem of Line-Integrals**, this is equal to $f(end) - f(start)$. The **start** is obtained by plugging in $t = 0$ into C:

$$start = (0^3, 0^2 + 1, 2(0) + 1) = (0,1,1) \quad .$$

The **end** is obtained by plugging in $t = 1$ into C:

$$end = (1^3, 1^2 + 1, 2(1) + 1) = (1,2,3) \quad ,$$

and the line-integral is $f(1,2,3) - f(0,1,1) = (1^2)(2^2)(3^2) - (0^2)(1^2)(1^2) = 36 - 0$.

Ans. to 1(b): 36.

Some Mistakes People Made: Do part (b) directly. Mathematically it is not a mistake, but it takes much longer. But even if done correctly, part (b) asked to use part (a), so you wouldn't get any credit for doing it directly, even if you get the correct answer.

2. Evaluate the line integral

$$\int_C y^2\,dx + x^2\,dy + xyz\,dz \quad ,$$

where $C : x = t^2\,,\; y = 3t\,,\; z = t^3\,,\; 0 \le t \le 1$.

Ans.: 61/10 [type number].

Sol. to 2.: This is a line-integral of the **vector-field** kind

$$\int_C P\,dx + Q\,dy + R\,dz \quad ,$$

where the input is a **vector field** (triple of functions) $\langle P, Q, R\rangle$ and a curve C (**not** to be confused with the arclengh kind $\int_C f(x,y,z)\,ds$, where the input is a **single** function $f(x,y,z)$ and a curve C).

Here

$$P = y^2 \quad , \quad Q = x^2 \quad , \quad R = xyz \quad ,$$

$$x = t^2 \quad , \quad y = 3t \quad , \quad z = t^3 \quad ,$$

$$dx = (2t)\,dt \quad , \quad dy = 3\,dt \quad , \quad dz = (3t^2)\,dt \quad ,$$

Plugging-everything in terms of t, we get

$$\int_C y^2\,dx + x^2\,dy + xyz\,dz =$$

$$\int_0^1 (3t)^2(2t)\,dt + (t^2)^2(3dt) + (t^2)(3t)(t^3)(3t^2)dt =$$

$$\int_0^1 [18t^3 + 3t^4 + 9t^8]dt = \left(18\frac{t^4}{4} + 3\frac{t^5}{5} + 9\frac{t^9}{9}\right)\Big|_0^1$$

$$= \left(\frac{9t^4}{2} + \frac{3t^5}{5} + t^9\right)\Big|_0^1 = \frac{9}{2} + \frac{3}{5} + 1 - 0 = \frac{61}{10} \quad .$$

Ans. to 2.: 61/10.

Some Mistakes People Made:

1. Confuse this line-integral with the other kind, and compute $|\mathbf{r}'(t)|$ which is **not** needed at all for the present kind.

2. Mess-up the plug-in and the algebra. Please check everything step. Surprisingly many did $(3t)^2(2t) = 6t$ or $(3t)^2(2t) = 6t^2$!. Of course it should be $(3t)(3t)(2t) = 18t^3$.

3. Minor, but extremely impolite "mistake". Most people, even the best-performing students, left the answer as 122/20. You should be able to realize that you can divide both top and bottom by 2 **even without a calculator**!

3. Evaluate

$$\int\int\int_E (5x^2+5y^2+5z^2)\,dV \quad ,$$

where E is bounded by the yz-plane and the hemispheres $x=-\sqrt{1-y^2-z^2}$ and $x=-\sqrt{4-y^2-z^2}$.

Ans.: 62π [type number].

Solution to 3.: Obviously we have to use **spherical** coordinates, first because the problem mentions hemi-**spheres** and second because the "phrase" $x^2+y^2+z^2$ should ring a bell: it is ρ^2 in spherical language.

We have two (hemi-) spheres here. The inner one has $\rho=1$ and the outer one has $\rho=2$ (since $\sqrt{4}=2$, some people by mistake too the bigger radius to be 4, watch out!.)

So the **default** description of the regin between the **spheres** is:

$$E=\{(\rho,\theta,\phi)\,|\,1\leq\rho\leq 2\,,\,0\leq\theta\leq 2\pi\,,\,0\leq\phi\leq\pi\} \quad .$$

But somehow we have to cut-it-in-half. Since the hemispheres are $x=-\sqrt{1-y^2-z^2}$ and $x=-\sqrt{4-y^2-z^2}$, it means that $x<0$ (in calclus square-roots are always positive), so this means that θ ranges from $\pi/2$ to $3\pi/2$ (and ϕ remains in its default) [For $x>0$, θ would be between $-\pi/2$ and $\pi/2$ (and ϕ remains in its default), for $y<0$, θ would be between π and 2π (and ϕ remains in its default), for $y>0$, θ would be between 0 and π (and ϕ remains in its default), for $z>0$, ϕ would be between 0 and $\pi/2$ (and θ remains in its default), for $z<0$, ϕ would be between $\pi/2$ and π (and θ remains in its default)].

So our region of integration is

$$E=\{(\rho,\theta,\phi)\,|\,1\leq\rho\leq 2\,,\,\pi/2\leq\theta\leq 3\pi/2\,,\,0\leq\phi\leq\pi\} \quad .$$

Using the 'dictionary' $x^2+y^2+z^2=\rho^2$ and $dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, we get that our integral is

$$\int\int\int_E (5x^2+5y^2+5z^2)\,dV \quad = \int\int\int_E 5(x^2+y^2+z^2)\,dV \quad =$$

$$\int_{\pi/2}^{3\pi/2}\int_0^{\pi}\int_1^2 5\rho^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_{\pi/2}^{3\pi/2}\int_0^{\pi}\int_1^2 5\rho^4\sin\phi\,d\rho\,d\phi\,d\theta \quad .$$

Since all the limits-of-integration are **numbers** and the integrand is a **product** of functions of single-variables, we can safely use the **separation trick** to get

$$\left(\int_{\pi/2}^{3\pi/2} d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_1^2 5\rho^4\,d\rho\right)$$

$$= \left(\theta\Big|_{\pi/2}^{3\pi/2}\right)\left(-\cos\phi\Big|_0^{\pi}\right)\left(\rho^5\Big|_1^2\right)$$

$$= (3\pi/2 - \pi/2)\,(-\cos\pi - -\cos 0)\left(2^5 - 1^5\right) = (\pi)(2)(31) = 62\pi \quad .$$

Ans.: 62π.

Mistakes people made: 1. Not realizing that $5x^2 + 5y^2 + 5z^2 = 5(x^2 + y^2 + z^2)$ and doing it the "long way" ($x = \rho\sin\phi\cos\theta$ etc.) and messing up. 2. Not setting the limits of integration correctly.

4. Evaluate the triple integral

$$\int\int\int_E 20yz\cos(x^5)\,dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,0 \leq x \leq 1\,,\, 0 \leq y \leq x\,,\, x \leq z \leq 2x\,\} \quad .$$

Ans.: $3\sin 1$ [type number].

Solution to 4.: We first write the triple-integral (alias volume-integral) as an **iterated-integral**

$$\int_0^1 \int_0^x \int_x^{2x} 20yz\cos(x^5)\,dz\ dy\,dx \quad .$$

The **inner**-integral is

$$\int_x^{2x} 20yz\cos(x^5)\,dz = 20y\cos(x^5)\int_x^{2x} z\,dz = 20y\cos(x^5)\frac{z^2}{2}\Big|_x^{2x}$$

$$= 10y\cos(x^5)((2x)^2 - x^2) = 10y\cos(x^5)(4x^2 - x^2) = 10y\cos(x^5)(3x^2) = 30yx^2\cos(x^5) \quad .$$

The **middle-integral** is

$$\int_0^x [AnswerToInnerIntegral]\,dy = x^2\cos(x^5)\int_0^x 30y\,dy$$

$$= x^2\cos(x^5)(15y^2\Big|_0^x) = x^2\cos(x^5)(15x^2 - 0) = x^2\cos(x^5)(15x^2 - 0) = 15x^4\cos(x^5) \quad .$$

The **outer-integral** (and the final answer) is:

$$\int_0^1 [AnswerToMiddleIntegral]\,dx = \int_0^1 15x^4\cos(x^5)\,dx \quad .$$

This is handled by the **substitution** $u = x^5$

$$\int_0^1 \cos(x^5)15x^4\,dx = 3\int_0^1 \cos(x^5)5x^4\,dx = 3\int_0^1 \cos(x^5)\,d(x^5) = 3\sin(x^5)\Big|_0^1 = 3\sin(1) - 3\sin(0) = 3\sin(1) \quad .$$

Ans.: $3\sin(1)$.

Warning: $\sin(1)$ is **not** 1, it is simply $\sin(1)$. If you had a calculator, you could find it in decimals, but you don't, so that's the final answer.

5. Find the Jacobian of the transformation from (x, y, z)-space to (u, v, w)-space.

$$x = uv \quad , \quad y = uw \quad , \quad z = vw.$$

Ans.: $-2uvw$ [type function].

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ w & 0 & u \\ 0 & w & v \end{vmatrix}$$

$$= v \begin{vmatrix} 0 & u \\ w & v \end{vmatrix} - u \begin{vmatrix} w & u \\ 0 & v \end{vmatrix} + 0 \begin{vmatrix} w & 0 \\ 0 & w \end{vmatrix}$$

$$= v(0v - uw) - u(wv - u0) + 0(w^2 - 0^2) = -2uvw \quad .$$

Ans.: $-2uvw$.

6. Evaluate the integral

$$\int\int_D 4e^{-2x^2-2y^2}\, dA \quad ,$$

where D is the region bounded by the semi-circle $y = -\sqrt{9-x^2}$ and the x-axis.

Ans.: $\pi(1-e^{-18})$ [type number]

Solution to 6.: This calls for **polar** coordinates, because of the "phrase" x^2+y^2 that in polar-speak is simply r^2 and because that the region concerns a semi-**circle**.

Since y is **negative** this is the **lower** semi-circle of radius 3 center the origin, so D is

$$D = \{(r,\theta)\,|\,0 \le r \le 3\,,\pi \le \theta \le 2\pi\}$$

Recall also that $dA = r\,dr\,d\theta$. So the integral is

$$\int_{\pi}^{2\pi}\int_0^3 4e^{-2r^2} r\,dr\,d\theta \quad .$$

By the **separation trick** this is

$$\left(\int_{\pi}^{2\pi} d\theta\right)\left(\int_0^3 4e^{-2r^2} r\,dr\right) \quad .$$

The first integral is simply $2\pi - \pi = \pi$. The second is handled via the **substitution** $u = -2r^2$:

$$\int_0^3 -e^{-2r^2}(-4r\,dr) = \int_0^3 -e^{-2r^2}\,d(-2r^2) = -e^{-2r^2}\Big|_0^3 \quad .$$

$$= -e^{-2(3)^2} - -e^0 = e^0 - e^{-18} = 1 - e^{-18}$$

Combining, we have:

Ans.: $\pi(1-e^{-18})$.

Note: It is impolite to leave e^0 like this. You should replace it by 1. Also, review substitution! Many people are rusty with it and try the wrong approach (for this problem), like integration-by-parts.

7. Sketch the region of integration and change the order of integration.

$$\int_0^2 \int_{4x}^8 F(x,y)\,dy\,dx$$

Ans.:

$$\int_0^8 \int_0^{y/4} F(x,y)\,dx\,dy$$

[type: double-integral of an abstract function]

Solution to 7.: The implied region of integration is

$$D = \{(x,y)\,|\,0 \le x \le 2\,,\, 4x \le y \le 8\}$$

and this is expressed as type-I style. This region has the **main-road** along the x-axis stretching from $x = 0$ to $x = 2$. The side-streets are parallel to the y-axis, starting at the line $y = 4x$ and going all the way up to the horizontal line $y = 8$. Sketching this (do it!) we get a triangle whose vertices are $(0,0),(0,8),(2,8)$. Writing this in type II-style we get

$$D = \{(x,y)\,|\,0 \le y \le 8\,,\, 0 \le x \le y/4\} \quad .$$

Using this as the basis for the iterated integral, we get: **Ans.**:

$$\int_0^8 \int_0^{y/4} F(x,y)\,dx\,dy \quad .$$

Very Common Mistake: People drew it kind-of-correctly, but then they chose the complementary triangle with vertices $(0,0),(2,0),(2,8)$.

8. Calculate the iterated integral

$$\int_1^2 \int_0^1 (6x + 6y^2)\, dx\, dy \quad .$$

Ans.: 17 [type number].

Solution to 8.: The **inner**-integral is

$$\int_0^1 (6x + 6y^2)\, dx = (3x^2 + 6y^2 x)\Big|_0^1 = (3(1)^2 + 6y^2(1)) - (3(0)^2 + 6y^2(0)) = 3 + 6y^2 \quad .$$

The **outside**-integral (and the final answer) is

$$\int_1^2 [AnswerToInnerIntegral] dy = \int_1^2 [3 + 6y^2] dy$$

$$= [3y + 2y^3]\Big|_1^2 = [3(2) + 2(2)^3] - [3(1) + 2(1)^3] = 6 + 16 - 5 = 17 \quad .$$

Ans.: 17.

Common Mistakes: Arithmetics. People mess-up simple addition and multiplication. Review your second-grade math!

9. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = 4x + 6y$ subject to the constraint $x^2 + y^2 = 13$.

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maximum value: 26 [type number]

minimum value: -26 [type number]

Solution to 9. Here $f = 4x + 6y$ and $g = x^2 + y^2$. $\bigtriangledown f = \langle 4, 6\rangle$, $\bigtriangledown g = \langle 2x, 2y\rangle$. Setting-up $\bigtriangledown f = \lambda \bigtriangledown g$ and spelling it out, and adding the **constraint equation** $x^2 + y^2 = 13$, we get the **system** of three equations and three unknowns:

$$4 = 2\lambda x \quad , \quad 6 = 2\lambda y \quad , \quad x^2 + y^2 = 13 \quad .$$

Dividing the second equation by the first, we get

$$\frac{6}{4} = \frac{2\lambda y}{2\lambda x} = \frac{y}{x}$$

So

$$\frac{y}{x} = \frac{3}{2} \quad .$$

and $y = \frac{3}{2}x$. Plugging this into the third equation we get:

$$x^2 + (\frac{3}{2}x)^2 = 13 \quad .$$

Simplifying:

$$\left(1 + \frac{9}{4}\right) x = 13 \quad ,$$

so

$$\left(\frac{13}{4}\right) x^2 = 13 \quad .$$

and $x^2 = 4$ which has **two solutions** $x = -2$ and $x = 2$. Going back to the relation $y = (3/2)x$, we get that

$x = -2$ gives $y = \frac{3}{2}(-2) = -3$

and

$x = 2$ gives $y = \frac{3}{2}(2) = 3$.

So the **finalists** are $(-2, -3)$ and $(2, 3)$. For the final contest, we plug-in into $f(x, y)$.

$f(-2, -3) = 4(-2) + 6(-3) = -8 - 18 = -26$ and $f(2, 3) = 4(2) + 6(3) = 8 + 18 = 26$. The largest of these is 26 and the smallest is -26.

Ans.: The maximum value is 26 and the minimum value is -26.

Very common mistake: Many people said that there are four finalists: in addition to the above two points also $(-2, 3)$ and $(2, -3)$. This is because if you do the same thing for y you would get $y = \pm 3$. But y and x are related via $y = (3/2)x$ so $(-2, 3)$ and $(2, -3)$ should **not** be considered.

[In this particular problem, considering them didn't affect the final answer, and even in general adding spurious contestants (that meet the constraint) will not change who are the champions, but technically it is a mistake.]

10. Find the local maximum and minimum **values** and saddle point(s) of the function $f(x, y) = (1 + xy)(x + y)$

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local maximum value(s): none

local minimum value(s): none

saddle point(s): $(-1, 1)$ and $(1, -1)$ [of type points]

Solution of 10.: Using algebra to expand, we have $f = x + y + x^2y + xy^2$.

$$f_x = 1 + 2xy + y^2 \quad , \quad f_y = 1 + 2xy + x^2 \quad .$$

For future reference:

$$f_{xx} = 2y \quad , \quad f_{xy} = 2x + 2y \quad , \quad f_{yy} = 2x \quad .$$

First we must solve the **system** of two equations and two unknowns $f_x = 0, f_y = 0$.

$$1 + 2xy + y^2 = 0 \quad , \quad 1 + 2xy + x^2 = 0 \quad .$$

At this stage many people got stumped. In this case the trick is to **subtract**, getting yet another equation:

$$y^2 - x^2 = 0 \quad .$$

Factorizing:

$$(y - x)(y + x) = 0 \quad .$$

This gives us **two options**: $y = x$ and $y = -x$.

Let's first explore the first option: $y = x$. Plugging into the first equation we get

$$1 + 2x^2 + x^2 = 0$$

$$1 + 3x^2 = 0$$

So $x^2 = -1/3$. But this has no solutions! So we have to **abandon** the first option.

Regarding the second option $y = -x$, plugging into the first equation gives:

$$1 - 2x^2 + x^2 = 0$$

So $x^2 = 1$ that has two solutions $x = -1$ and $x = 1$. Since right now $y = -x$, this means that when $x = -1$, $y = -(-1) = 1$ and when $x = 1$, $y = -1$. So we have **two** critical points: $(-1, 1)$ and $(1, -1)$.

At $(-1,1)$, $f_{xx}=2, f_{xy}=0, f_{yy}=-2$, so $D=(2)(-2)-0^2=-4<0$ which means that $(-1,1)$ is a **saddle-point**.

At $(1,-1)$, $f_{xx}=-2, f_{xy}=0, f_{yy}=2$, so $D=(-2)(2)-0^2=-4<0$ which means that $(1,-1)$ is a **saddle-point**.

Ans.: There are no maximum values and no miminum values; The saddle points are $(-1,1)$ and $(1,-1)$.