When the coefficients are constant functions (or constants for short), then our homogeneous linear diff.eq. of order \( n \) looks as follows

\[
a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + \ldots + a_{n-1} y'(x) + a_n y(x) = 0 ,
\]

for some numbers \( a_0, a_1, \ldots, a_n \).

To solve it, you form the characteristic equation, by replacing \( y(x) \) by 1, \( y'(x) \) by \( r \), \( y''(x) \) by \( r^2 \), \( y'''(x) \) by \( r^3 \) etc..

Then you solve it (either by inspection, or with Maple or Matlab).

**Case 1: Real and Distinct Roots**, let’s call them \( r_1, r_2, \ldots, r_n \). Then the general solution is

\[
y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} ,
\]

where \( c_1, c_2, \ldots, c_n \) are arbitrary constants.

**Case 2: One or more Root is repeated**: If the root \( r_1 \) is a double root then the corresponding solutions are \( e^{r_1 x}, x e^{r_1 x} \). If the root \( r_1 \) is a triple root then the corresponding solutions are \( e^{r_1 x}, x e^{r_1 x}, x^2 e^{r_1 x} \). If the root \( r_1 \) is a quadruple root the the corresponding solutions are \( e^{r_1 x}, x e^{r_1 x}, x^2 e^{r_1 x}, x^3 e^{r_1 x} \). Etc.

**Case 3: Complex Roots**: For every pair \( \lambda \pm i\mu \) correspond the solutions \( e^{\lambda x} \cos \mu x \) and \( e^{\lambda x} \sin \mu x \).

**Problem 15.1**: Find the general solution of

\[
a. \ y'''(t) - y'(t) = 0 \\
b. \ y'''(t) - 6y''(t) + 11y(t) - 6y(t) = 0
\]

**Solution of 15.1a**: The characteristic equation is \( r^3 - r = 0 \). Factorizing, we get \( r(r^2 - 1) = 0 \) and then \( r(r-1)(r+1) = 0 \), so the three roots are distinct and they are \( r = -1, \ r = 0, \ \text{and} \ r = 1 \) corresponding to the fundamental solutions are \( e^{-t}, e^0 = 1, e^t \) and the general solution is a generic linear combination

\[
y(t) = c_1 e^{-t} + c_2 + c_3 e^t .
\]

**Ans. to 15.1a**: \( y(t) = c_1 e^{-t} + c_2 + c_3 e^t \).

**Solution of 15.1b**: The characteristic equation is \( r^3 - 6r^2 + 11r - 6 = 0 \). Factorizing, we get \( (r-1)(r-2)(r-3) = 0 \) and so the three roots are distinct and they are \( r = 1, \ r = 2, \ \text{and} \ r = 3 \) corresponding to the fundamental solutions are \( e^t, e^{2t}, e^{3t} \) and the general solution is a generic linear combination

\[
y(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} .
\]
Ans. to 15.1b \( y(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \)

**Problem 15.2:** Find the general solution of

a. \( y''' - 2y''(x) + y'(x) = 0 \)

b. \( y^{(4)} - 3y^{(3)}(x) + 3y''(x) - y'(x) = 0 \)

**Solution of 15.2a:** The characteristic equation is \( r^3 - 2r^2 + r = 0 \). Factorizing, we get \( r(r - 1)^2 = 0 \) and so the root \( r = 0 \) is a single-root, but the root \( r = 1 \) is a double-root. So the fundamental solutions are \( e^{0t} = 1, e^t, te^t \), and the general solution is a generic linear combination

\[
y(t) = c_1 + c_2 e^t + c_3 t e^t .
\]

Ans. to 15.2a: \( y(t) = c_1 + c_2 e^t + c_3 t e^t . \)

**Solution of 15.2b:** The characteristic equation is \( r^4 - 3r^3 + 3r^2 - r = 0 \). Factorizing, we get \( r(r - 1)^3 = 0 \) and so the root \( r = 0 \) is a single-root, but the root \( r = 1 \) is a triple-root. So the solutions are \( e^{0t} = 1, e^t, te^t, t^2 e^t \), and the general solution is a generic linear combination

\[
y(t) = c_1 + c_2 e^t + c_3 te^t + c_4 t^2 e^t .
\]

Ans. to 15.2b: \( y(t) = c_1 + c_2 e^t + c_3 te^t + c_4 t^2 e^t . \)

**Problem 15.3:** Find the general solution of

a. \( y^{(4)}(x) - y(x) = 0 \)

b. \( y^{(4)} + 2y''(x) + y(x) = 0 \)

**Solution of 15.3a:** The characteristic equation is \( r^4 - 1 = 0 \). Factorizing, we get \((r^2 - 1)(r^2 + 1) = 0\), and then \((r - 1)(r + 1)(r^2 + 1) = 0\). and so the root \( r = -1, r = 1 \) and \( r = \pm i = 0 \pm 1 \cdot i \). So the solutions are \( e^{-t}, e^t, e^{0t} \cos t, e^{0t} \sin t \), so they are \( e^{-t}, e^t, \cos t, \sin t \), and the general solution is a generic linear combination

\[
y(t) = c_1 e^{-t} + c_2 e^t + c_3 \cos t + c_4 \sin t .
\]

Ans. to 15.3a: \( y(t) = c_1 e^{-t} + c_2 e^t + c_3 \cos t + c_4 \sin t . \)

**Solution of 15.3b:** The characteristic equation is \( r^4 + 2r^2 + 1 = 0 \). Factorizing, we get \((r^2 + 1)^2 = 0\), and so the roots are: \( r = \pm i = 0 \pm 1 \cdot i \), each repeated twice! So the solutions are \( \cos t, \sin t, t \cos t, t \sin t \), and the general solution is a generic linear combination

\[
y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t
\]
Ans. to 15.3a: $y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$

Problem 15.1a': Solve the initial value problem

a. $y''(t) - y'(t) = 0$ ; $y(0) = 3$ , $y'(0) = 0$ , $y''(0) = 2$

Solutions of 15.1a': We first do 15.1a getting that the general solution is

$$y(t) = c_1 e^{-t} + c_2 + c_3 e^t .$$

We now find expressions for $y'(t)$ and $y''(t)$:

$$y'(t) = -c_1 e^{-t} + c_3 e^t ,
\quad y''(t) = c_1 e^{-t} + c_3 e^t .$$

We now plug-in $t = 0$

$$y(0) = c_1 + c_2 + c_3 ,
\quad y'(0) = -c_1 + c_3 ,
\quad y''(0) = c_1 + c_3 .$$

We now use the initial conditions, $y(0) = 3, y'(0) = 0, y''(0) = 2$ getting the system of linear equations

$$c_1 + c_2 + c_3 = 3 ,
\quad -c_1 + c_3 = 0 ,
\quad c_1 + c_3 = 2 ,$$

whose solution is $c_1 = 1, c_2 = 1, c_3 = 1$. We now go back to the general solution and substitute for these.

$$y(t) = 1 \cdot e^{-t} + 1 + 1 \cdot e^t = e^{-t} + 1 + e^t .$$

Ans. to 15.1a': $y(t) = e^{-t} + 1 + e^t$. 

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