So far we considered one diff.eq. with one unknown function to be found, usually written \( y(t) \) or \( y(x) \), where \( t \) or \( x \) were the independent variable and \( y \) was the dependent variable.

Often, in applications, we have several (say \( n \)) differential equations with several (usually the same number, \( n \)) of unknown functions, called \( x_1(t), x_2(t), \ldots, x_n(t) \). We only consider first order equations, i.e. when we do systems, we only have the first derivative show up.

The general format of a System of First-Order Differential Equations with \( n \) functions to look for, \( x_1(t), \ldots, x_n(t) \) is

\[
\begin{align*}
x'_1(t) &= F_1(t, x_1(t), x_2(t), \ldots, x_n(t)) , \\
x'_2(t) &= F_2(t, x_1(t), x_2(t), \ldots, x_n(t)) , \\
& \quad \vdots \\
& \quad \vdots \\
x'_n(t) &= F_n(t, x_1(t), x_2(t), \ldots, x_n(t)) .
\end{align*}
\]

Here \( F_1(t, x_1, \ldots, x_n), \ldots, F_n(t, x_1, \ldots, x_n) \) are some (possibly very complicated) multivariable functions of \( n + 1 \) arguments.

If we specify initial conditions

\[
x_1(t_0) = x_1^0 , \quad x_2(t_0) = x_2^0 , \quad \ldots , \quad x_n(t_0) = x_n^0 ,
\]

then we have an initial value problem.

Of course, it is usually not possible to get an exact solution, in terms of a formula, and the best that we can hope for is to find good approximations, on the computer, but, in an abstract sense, we know that solutions exist, if the functions \( F_1, F_2, \ldots, F_n \) featured in the system, are not too crazy.

We have


If the functions \( F_1, \ldots, F_n \) and all their partial derivatives are continuous (do not blow up and have no breaks) in a box-like region \( R \) of the \( (n + 1) \) dimensional \( t x_1 \ldots x_n \) space containing the point \( (t_0, x_1^0, \ldots, x_n^0) \). Then there is an interval \(|t - t_0| < h\) in which there is unique solution of the above initial value problem.
An important special case of systems of Diff.Eqs. are **Linear Systems of Diff.Eq.s.** whose format is

\[
x'_1(t) = p_{11}(t)x_1(t) + \ldots + p_{1n}(t)x_n + g_1(t) \\
x'_2(t) = p_{21}(t)x_1(t) + \ldots + p_{2n}(t)x_n + g_2(t) \\
\ldots \\
\ldots \\
x'_n(t) = p_{n1}(t)x_1(t) + \ldots + p_{nn}(t)x_n + g_n(t) .
\]

If all the \(g_i(t)\) are 0 then we have a **homogeneous system**.

If all the coefficient functions \(p_{ij}(t)\) are continuous in an interval \(I\), then we are guaranteed a solution satisfying any initial conditions.

**Converting ONE Higher-Order Diff.Eq. to a FIRST-ORDER System**

Whenever we have one diff.eq. of the format

\[
y^{(n)}(t) = F(t, y(t), y'(t), \ldots, y^{(n-1)}(t)) ,
\]

there is a quick way to make it into a first order system, as follows. Assuming that we already know \(y(t)\), we define

\[
x_1(t) = y(t) , \quad x_2(t) = y'(t) , \quad x_3(t) = y''(t) , \quad \ldots , \quad x_n(t) = y^{(n-1)}(t),
\]

then

\[
x'_1(t) = x_2(t) \\
x'_2(t) = x_3(t) \\
x'_3(t) = x_4(t) \\
\ldots \\
x'_{n-1}(t) = x_n(t) \\
x'_n(t) = F(t, x_1, \ldots, x_n) .
\]

Note that only the last equation is “complicated”, the first \(n - 1\) ones are very simple.

**Problem 17.1:** Convert the following third-order diff.eq. to a system of first-order diff.eq.s

\[
y'''(t) = \cos(y''(t) + t^3 + y'(t)y(t)) + \sin(y(t))
\]

**Solutions to 17.1:** The first \(n - 1\) equations are **always** the same. Here \(n = 3\) (since it is a third-order diff.eq.) so our first two equations are

\[
x'_1(t) = x_2(t)
\]
\[ x'_2(t) = x_3(t) \]
and to get the last one, you replace \( y'''(t) \) by \( x'_2(t) \) and \( y''(t) \) by \( x_3(t) \), \( y'(t) \) by \( x_2(t) \), and \( y(t) \) by \( x_1(t) \). In this problem
\[ x'_3(t) = \cos(x_3(t) + t^3 + x_2(t)x_1(t)) + \sin(x_1(t)) \]

**Ans. to 17.1:**
\[
\begin{align*}
x'_1(t) &= x_2(t) , \\
x'_2(t) &= x_3(t) , \\
x'_3(t) &= \cos(x_3(t) + t^3 + x_2(t)x_1(t)) + \sin(x_1(t)).
\end{align*}
\]

But sometimes one can go the other way. Given a first-order system, we can solve it using what we know about higher-order diff.eq. Lucky for us, we only need to do it for linear systems with constant coefficients.

**Problem 17.2** Solve the initial value problem for the system
\[
\begin{align*}
x'_1(t) &= -2x_1(t) + x_2(t) , \\
x'_2(t) &= x_1(t) - 2x_2(t) ; \\
x_1(0) &= 2 , \\
x_2(0) &= 3 .
\end{align*}
\]

**Solution to 17.2**

**Step 1:** Use the first equation, and algebra, to express \( x_2(t) \) in terms of \( x_1(t) \) (and its derivative \( x'_1(t) \)).
\[
x_2(t) = x'_1(t) + 2x_1(t)
\]

**Step 2:** Substitute this into second equation:
\[
(x'_1(t) + 2x_1(t))^' = x_1(t) - 2(x'_1(t) + 2x_1(t))
\]

**Step 3:** Use calculus and algebra to simplify
\[
x''_1(t) + 2x'_1(t) = x_1(t) - 2x'_1(t) - 4x_1(t)
\]
\[
x''_1(t) + 4x'_1(t) + 3x_1(t) = 0
\]

**Step 4:** Go back to Step 1 and plug-in \( t = 0 \) and use algebra to find \( x'_1(0) \):
\[
x_2(0) = x'_1(0) + 2x_1(0) .
\]
\[
x'_1(0) = x_2(0) - 2x_1(0) = 3 - 2 \cdot 2 = 3 - 4 = -1 .
\]

**Step 5:** Solve the initial value problem
\[
x''_1(t) + 4x'_1(t) + 3x_1(t) = 0 , \\
x_1(0) = 2 , \\
x'_1(0) = -1
\]
We get \( x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \)

**Step 6:** Go back to Step 1 and find out what is \( x_2(t) \):

\[
x_2(t) = x_1'(t) + 2x_1(t) = \left( \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \right)' + 2\left( \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \right) = -\frac{5}{2}e^{-t} + \frac{3}{2}e^{-3t} + 5e^{-t} - e^{-3t} = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}
\]

**Ans. to 17.2:** \( x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \), \( x_2(t) = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t} \).

**Review of Vectors and Matrices**

Look it up in wikipedia. In Maple you use the package LinearAlgebra. Look up the commands Matrix, Inverse, Multiply.