1. (10 pts.) Find the general solution to the following differential equation

\[ y''(t) + 100y(t) = 0. \]

**Ans.:** \( y(t) = c_1 \sin 10t + c_2 \cos 10t \) (where \( c_1, c_2 \) are arbitrary constants)

The **characteristic equation** is \( r^2 + 100 = 0 \). Solving it we get \( r = \pm \sqrt{-100} = \pm \sqrt{100} \sqrt{-1} = 0 \pm 10i \).

So \( \lambda = 0 \) and \( \mu = 10 \). Plugging into the general formula

\[ y(t) = e^{\lambda t}(c_1 \sin \mu t + c_2 \cos \mu t), \]

we get

\[ y(t) = e^{0 \cdot t}(c_1 \sin 10t + c_2 \cos 10t) = 1 \cdot (c_1 \sin 10t + c_2 \cos 10t) = c_1 \sin 10t + c_2 \cos 10t. \]

**Note:** Some people added “steady oscillation”. This is true, but I never asked what kind of oscillations it is. This time I was nice, and didn’t take any points off, but next time, if I will ask you “Who is the president of the USA?” and you would answer “Mr. Obama is the president, and Mr. Biden is the vice-president’ I will take points off. Please answer what has been asked! Not less, but also not more!
2. (10 pts.) Find the Wronskian, \( W(f(t), g(t)) \) of the following pair of functions:

\[
f(t) = e^{3t}, \quad g(t) = te^{3t}.
\]

Ans.: 

\[
W(f(t), g(t)) = e^{6t}.
\]

\[
W(f(t), g(t)) = f(t)g'(t) - f'(t)g(t).
\]

Here \( f(t) = e^{3t} \) so \( f'(t) = 3e^{3t} \).

Also \( g(t) = te^{3t} \). By the **product rule**, followed by the **chain rule**

\[
g'(t) = t' e^{3t} + t(e^{3t})' = 1 \cdot e^{3t} + t(3e^{3t}) = e^{3t}(1 + 3t).
\]

So we have

\[
W(f(t), g(t)) = e^{3t} \cdot e^{3t}(1 + 3t) - 3e^{3t} \cdot te^{3t} = e^{6t}(1 + 3t) - 3te^{6t}
\]

\[
= (1 + 3t - 3t)e^{6t} = 1 \cdot e^{6t} = e^{6t}.
\]

Comment: Most people got it right, but some people forgot (or never knew) how to apply the product and chain rules. More depressingly, some people have trouble simplifying \( e^{3t} \cdot e^{3t} \).

This is **basic algebra**. If you are having trouble, please review this! It is much more important than differential equations!
3. (10 pts.) Solve the initial value problem
\[ y''(t) - 3y'(t) = 0, \quad y(0) = 2, \quad y'(0) = 3. \]

**Ans.:** \[ y(t) = 1 + e^{3t}. \]

The **characteristic equation** is
\[ r^2 - 3r = 0. \]

For this one it is much easier to factorize than to use the quadratic formula (that some people did and messed up!).
\[ r(r - 3) = 0. \]
Which is the same as
\[ (r - 0)(r - 3) = 0. \]
So we have **two** distinct real roots \( r_1 = 0 \) and \( r_2 = 3 \). The general solution in this case is
\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \]
so in our case it is
\[ y(t) = c_1 e^{0t} + c_2 e^{3t} = c_1 + c_2 e^{3t}. \]

Now it is time to find the constants by using the **initial conditions** \( y(0) = 2 \) and \( y'(0) = 3 \). For future reference
\[ y'(t) = 3c_2 e^{3t}, \]
so
\[ y(0) = c_1 + c_2, \quad y'(0) = 3c_2. \]

Taking advantage of the initial conditions we get the system of two linear equations with two unknowns, \( c_1, c_2 \):
\[ c_1 + c_2 = 2, \quad 3c_2 = 3. \]
From the second we get that \( c_2 = 3/3 = 1 \). Plugging into the first equation, we get \( c_1 + 1 = 2 \) so \( c_1 = 1 \). Going back to the general solution \( y(t) = c_1 + c_2 e^{3t} \), and substituting \( c_1 = 1, c_2 = 1 \) we get
\[ y(t) = 1 + 1 \cdot e^{3t} = 1 + e^{3t}. \]
4. (10 pts.) Use the Euler method to find an approximate value for $y(1.2)$ if $y(x)$ is the solution of the initial value problem differential equation

$$y' = x + y \quad , \quad y(1) = 0 \quad ,$$

using mesh-size $h = 0.1$.

**Reminder:** $x_n = x_0 + nh$, $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$.

**Ans.:** $y(1.2)$ is approximately equal to: 0.22 (or $\frac{11}{50}$).

Here $x_0 = 1$ (since the initial condition is $y(1) = 0$ and the argument of $y$ is 1). Also $h = 0.1$, so

$$x_0 = 1 \quad , \quad x_1 = 1.1 \quad , \quad x_2 = 1.2 \quad .$$

Also $y_0 = 0$ (since the initial condition is $y(1) = 0$ and the right side is 0). In this problem

$$f(x, y) = x + y \quad .$$

Using $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, with $n = 1$ we have

$$y_1 = y_0 + (0.1) \cdot f(x_0, y_0) = 0 + 0.1 \cdot f(1, 0) = 0.1 \cdot (1 + 0) = 0.1 \quad .$$

Using $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, with $n = 2$ we have

$$y_2 = y_1 + (0.1) \cdot f(x_1, y_1) = 0.1 + 0.1 \cdot f(1.1, 0.1) = 0.1 + 0.1 \cdot (1.1 + 0.1) = 0.1 + 0.1 \cdot (1.2) = 0.1 + 0.12 = 0.22 \quad .$$

This is the answer.

**Comment:** Most people got it right, but some people went ‘over and above the call of duty’ and went one more step with $n = 3$, and got an approximation to $y(1.3)$. They got only 5 points out of 10, and even this is charity. You have to answer what I asked for!
5. (10 pts.) For the following first-order differential equation, decide whether or not it is exact. If it is, solve it. Leave the answer in implicit format.

\[(3x^2 + y) + (x + 2y) y' = 0\]

Ans.: \(x^3 + xy + y^2 = C\) (where \(C\) is an arbitrary constant).

\[
M = 3x^2 + y \quad , \quad N = x + 2y \quad .
\]

\[
M_y = 1 \quad , \quad N_x = 1 \quad ,
\]

so \(M_y = N_x\) and the diff.eq. is indeed exact.

\[
F(x, y) = \int M(x, y) \, dx = \int (3x^2 + y) \, dx = x^3 + xy + \phi(y)
\]

where \(\phi(y)\) is a function of \(y\) alone yet TBD. Using \(F_y = N\) we get

\[
x + \phi'(y) = x + 2y \quad .
\]

Thanks to algebra:

\[
\phi'(y) = 2y \quad .
\]

Integrating with respect to \(y\):

\[
\phi(y) = \int (2y) \, dy = y^2 \quad .
\]

Going back to \(F(x, y)\) we get

\[
F(x, y) = x^3 + xy + y^2 \quad .
\]

**THIS IS NOT THE FINAL ANSWER.** People who put this, got at most five out of the ten points. The Final answer is \(F(x, y) = C\), where \(C\) is an arbitrary constant.
6. (10 pts.) For the following diff. eq. determine the critical (equilibrium) solutions and decide, for each such solution, whether it is asymptotically stable, unstable, or semi-stable.

\[ \frac{dy}{dt} = y^2 - 3y \ . \]

Ans.: \( y = 0 \), asymptotically stable ; \( y = 3 \), asymptotically unstable .

This is an autonomous diff.eq. Setting the right side equal to 0

\[ y^2 - 3y = 0 \ , \]

and solving

\[ y(y - 3) = 0 \ , \]

we get two equilibrium solutions \( y = 0 \) and \( y = 3 \). Let’s investigate them each.

For \( y = -.1 \), \( y' = (-0.1)(-0.1 - 3) \) is positive so it tends to go up. For \( y = .1 \), \( y' = (0.1)(0.1 - 3) \) is negative so it tends to go down. So in either case it tends to go towards \( y = 0 \) and it is stable.

For \( y = 2.9 \), \( y' = (2.9)(2.9 - 3) \) is negative so it tends to go down. For \( y = 3.1 \), \( y' = (3.1)(3.1 - 3) \) is positive so it tends to go up. So in either case it tends to away from \( y = 3 \) and it is unstable.
7. (10 pts.) Find the maximal open interval for which the following first-order differential equation and initial value problem is guaranteed to have a unique solution. Explain!

\[(t - 3)(t + 4) y'(t) + e^t y(t) = t^2, \quad y(-1) = 10.\]

**Ans.:** \(-4 < t < 3\).

Since this is an initial value problem, we are only interested in what is going on near \(t = -1\). Dividing by the coefficient of \(y'(t)\) we get:

\[y'(t) + \frac{e^t}{(t - 3)(t + 4)} y(t) = \frac{t^2}{(t - 3)(t + 4)}, \quad y(-1) = 10.\]

The coefficient of \(y(t)\) and the right side blow-up at \(t = -4\) and at \(t = 3\) (since then we have division by 0), but as long as you stay in the interval \(-4 < t < 3\) things are nice and calm. So the maximal interval is indeed \(-4 < t < 3\).

**Comment:** Some people gave as the answer the three intervals

\[-\infty < t < -4, \quad -4 < t < 3, \quad 3 < t < \infty.\]

This is the right answer to a different question: “Find the maximal open intervals (in plural!) for which there are unique solutions to the differential equation \((t-3)(t+4)y'(t) + e^t y(t) = t^2\) (where no initial condition is given).” Don’t confuse the two kinds of problems! I was being nice this time and gave 6 out of 10 points for the people who gave this answer. I won’t be as nice next time!
8. (10 pts.) Find an equation of the curve that passes through the point \((1, 2)\) and whose slope at \((x, y)\) is \(x/y\).

**Ans.:** \(y^2 - x^2 = 3\) or \(y = \sqrt{x^2 + 3}\).

Since **slope** is **derivative**, we have to solve the diff.eq.

\[
\frac{dy}{dx} = \frac{x}{y}.
\]

Since the curve should pass through the point \((1, 2)\) we need to really solve the IVP

\[
\frac{dy}{dx} = \frac{x}{y}, \quad y(1) = 2.
\]

The diff.eq. is handled via the method of **separation of variables**.

\[
\int y \, dy = \int x \, dx.
\]

Doing the integration:

\[
y^2 = \frac{x^2}{2} + C.
\]

Multiplying by 2:

\[
y^2 = x^2 + C.
\]

(since \(2C = C\)). Now we plug-in \(x = 1\) \(y = 2\) and find out what \(C\) is:

\[
2^2 = 1^2 + C.
\]

So

\[
C = 4 - 1 = 3.
\]

Going back to the general solution we get

\[
y^2 = x^2 + 3,
\]

and a little bit nicer, in **implicit** format \(y^2 - x^2 = 3\) (BTW this is a hyperbola). This is acceptable as a final answer, since this is a curve (with two components, but that’s OK). But some people love **explicit** answers and took the square-root, getting

\[
y = \pm \sqrt{x^2 + 3}.
\]

That’s very nice of them. Indeed they are good students, and remember the solution of \(x^2 = a\) is \(\pm \sqrt{a}\). But the curve \(y = -\sqrt{x^2 + 3}\) does not pass through the point \((1, 2)\) so it must be discarded. So those people who insist on explicit answers, should have had \(y = \sqrt{x^2 + 3}\) without the ±.

The irony is that people who are not-so-good-students, and forgot about the ± would have gotten it completely right.

I only took two points off for having ±.
9. (10 pts.) Solve the initial value problem

\[ y'(t) - 3y(t) = e^{2t} , \quad y(0) = 1. \]

**Ans.:** \( y(t) = 2e^{3t} - e^{2t} \).

We use the method of **integrating factor**.

\[ p(t) = -3 \ , \]

so

\[ I(t) = e^{\int p(t) \, dt} = e^{\int -3 \, dt} = e^{-3t} \ . \]

Multiplying the diff. eq. by \( I(t) = e^{-3t} \) we get:

\[ e^{-3t}y'(t) - 3e^{-3t}y(t) = e^{2t} \cdot e^{-3t} = e^{-t} \ . \]

The left side is **guaranteed** to be \((I(t)y(t))'\) (but it is a good idea to check), so

\[ (e^{-3t}y(t))' = e^{-t} \ . \]

Integrating:

\[ e^{-3t}y(t) = \int e^{-t} \, dt = -e^{-t} + C \ . \]

**Warning:** Don’t forget the +C!

Dividing by \( e^{-3t} \) we get

\[ y(t) = \frac{-e^{-t} + C}{e^{-3t}} = -e^{2t} + Ce^{3t} \ . \]

Now it is time to use the **initial condition**, \( y(0) = 1 \).

\[ y(0) = -e^{2 \cdot 0} + Ce^{3 \cdot 0} = -1 + C \ . \]

But \( y(0) = 1 \) so

\[ 1 = -1 + C \ . \]

Solving for \( C \), we get \( C = 2 \). Finally going back to the general solution \( y(t) = -e^{2t} + Ce^{3t} \) and pugging-in \( C = 2 \) we get the final answer

\[ y(t) = -e^{2t} + 2e^{3t} \ . \]
10. (10 pts.) Decide whether \( y(t) = te^{2t} \) is a solution of the initial value differential equation

\[
y''(t) - 4y'(t) + 4y(t) = 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1 \quad .
\]

Explain everything!

Comment: This is a problem from Lecture 1. It could also be done by actually solving it, using the method of Lecture 11 (that was not part of this exam). People who did it correctly got full credit, but they were really supposed to do it the Assignment 1 way, since it says ‘decide’ not solve!

\[
y(t) = te^{2t}
\]

By the product and chain rules:

\[
y'(t) = (te^{2t})' = t'e^{2t} + t(e^{2t})' = 1 \cdot e^{2t} + t(2e^{2t}) = (1 + 2t)e^{2t} \quad .
\]

\[
y''(t) = ((1 + 2t)e^{2t})' = (1 + 2t)'e^{2t} + (1 + 2t)(e^{2t})' \quad = 2e^{2t} + (1 + 2t)(2e^{2t}) \quad = (4 + 4t)e^{2t} \quad .
\]

Going to the diff.eq. and simplifying, we get

\[
y''(t) - 4y'(t) + 4y(t) \quad = (4 + 4t)e^{2t} - 4(1 + 2t)e^{2t} + 4te^{2t} \quad = (4 + 4t - 4 - 8t + 4t)e^{2t} \quad = 0 \cdot e^{2t} \quad = 0
\]

Yea! we got something right, so the proposed function is indeed a solution of the diff.eq.

We still have to worry about the initial conditions.

\[
y(0) = 0 \cdot e^{2 \cdot 0} = 0
\]

\[
y'(0) = (1 + 2 \cdot 0)e^{2 \cdot 0} = 1
\]

So the initial conditions are also OK!