1. Solve the initial value problem
\[ y^{(4)}(t) - 5y''(t) + 4y(t) = 0 \quad ; \quad y(0) = 4 \quad , \quad y'(0) = 0 \quad , \quad y''(0) = 2 \quad , \quad y'''(0) = 8 \].

**Sol. to 1:** We first find the General Solution. The characteristic equation is
\[ r^4 - 5r^2 + 4 = 0 \].

Factorizing (treating \( r^2 \) as a)
\[ (r^2 - 1)(r^2 - 4) = 0 \].

Factorizing as much as possible:
\[ (r + 1)(r - 1)(r + 2)(r - 2) = 0 \].

So we have four distinct real roots \(-2, -1, 1, 2\), and the General Solution is
\[ y(t) = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^t + c_4 e^{2t} \].

Where \( c_1, c_2, c_3, c_4 \) are arbitrary constants.

**Note:** Most people got this part correctly.

Now comes the ‘fun’ part. **First**, we need \( y'(t), y''(t), y'''(t) \).
\[ y'(t) = -2c_1 e^{-2t} - c_2 e^{-t} + c_3 e^t + 2c_4 e^{2t} \].
\[ y''(t) = 4c_1 e^{-2t} + c_2 e^{-t} + c_3 e^t + 4c_4 e^{2t} \].
\[ y'''(t) = -8c_1 e^{-2t} - c_2 e^{-t} + c_3 e^t + 8c_4 e^{2t} \].

Plug-in \( t = 0 \):
\[ y(0) = c_1 + c_2 + c_3 + c_4 \].
\[ y'(0) = -2c_1 - c_2 + c_3 + 2c_4 \].
\[ y''(0) = 4c_1 + c_2 + c_3 + 4c_4 \].
\[ y'''(0) = -8c_1 - c_2 + c_3 + 8c_4 \].

Now we use the initial conditions, to get a system of four linear equations with four unknowns \( c_1, c_2, c_3, c_4 \)
\[ c_1 + c_2 + c_3 + c_4 = 4 \].
Now comes the really fun part!

One way is to use Gaussian elimination using row-operations, but we are not supposed to know that. So we will use back substitution.

From the first equation we get

\[ c_4 = 4 - c_1 - c_2 - c_3 \]

Let’s plug this into the last three equations.

\[-2c_1 - c_2 + c_3 + 2(4 - c_1 - c_2 - c_3) = 0 \]
\[4c_1 + c_2 + c_3 + 4(4 - c_1 - c_2 - c_3) = 2 \]
\[-8c_1 - c_2 + c_3 + 8(4 - c_1 - c_2 - c_3) = 8 \]

Open-up parentheses!

\[-2c_1 - c_2 + c_3 + 8 - 2c_1 - 2c_2 - 2c_3 = 0 \]
\[4c_1 + c_2 + c_3 + 16 - 4c_1 - 4c_2 - 4c_3 = 2 \]
\[-8c_1 - c_2 + c_3 + 32 - 8c_1 - 8c_2 - 8c_3 = 8 \]

Clean-up!

\[-4c_1 - 3c_2 - c_3 = -8 \]
\[-3c_2 - 3c_3 = -14 \]
\[-16c_1 - 9c_2 - 7c_3 = -24 \]

Now we have to solve a system of three equations with three unknowns, \(c_1, c_2, c_3\). Since the second equation does not have \(c_1\), it is most efficient to take this as the ‘pivot’. we get

\[-3c_2 = -14 + 3c_3 \]
Hence
\[ c_2 = \frac{14}{3} - c_3 \ . \]

Now we plug this into the first and third equations:
\[
-4c_1 - 3\left(\frac{14}{3} - c_3\right) - c_3 = -8 \ , \\
-16c_1 - 9\left(\frac{14}{3} - c_3\right) - 7c_3 = -24 .
\]

Open-up parentheses:
\[
-4c_1 - 14 + 3c_3 - c_3 = -8 \ , \\
-16c_1 - 42 + 9c_3 - 7c_3 = -24 .
\]

Clean-up!
\[
-4c_1 + 2c_3 = 6 \ , \\
-16c_1 + 2c_3 = 18 .
\]

Now we only have to solve a system of two equations and two unknowns, \(c_1, c_3\).

From the first equation
\[ 2c_3 = 6 + 4c_1 \ , \]
divide by 2:
\[ c_3 = 3 + 2c_1 \ , \]
Substitute into the second equation
\[ -16c_1 + 2(3 + 2c_1) = 18 . \]

Open-up parentheses (aka as expand):
\[ -16c_1 + 6 + 4c_1 = 18 . \]

Clean-up:
\[ -12c_1 = 12 \]
Divide by \(-12\), we get
\[ c_1 = -1 \ . \]

Yea, we found one of these four guys. Now it is time to backtrack, by doing back-substitution.
\[ c_3 = 3 + 2c_1 = 3 + 2(-1) = 1 \ , \]
So \(c_3 = 1\).

Two more to go!
\[ c_2 = \frac{14}{3} - c_1 = \frac{14}{3} - 1 = \frac{11}{3} \]
And last but not least!

\[ c_4 = 4 - c_1 - c_2 - c_3 = 4 - (-1) - \frac{11}{3} - 1 = 4 - \frac{11}{3} = \frac{1}{3}. \]

Summarizing: \( c_1 = -1, \) \( c_2 = \frac{11}{3}, \) \( c_3 = 1, \) \( c_4 = \frac{1}{3}. \)

FINALLY, we go back to the General Solution, and substitute these values.

**Ans. to 1:**

\[ y(t) = -e^{-2t} + \frac{11}{3} e^{-t} + e^t + \frac{1}{3} e^{2t}. \]

**Comments:** 1. Most people tackled it the right way, but because it was so long, either didn’t even try, or tried, but sooner or later messed up. No one got the right answer.

2. Such a complicated problem is not likely to show up on an exam. In fact, I am almost sure, that I ‘cooked’ a problem with a much nicer answer, but I entered the initial conditions differently, and I changed a relatively (computationally) easy problem to a nightmare. Sorry. Then again, it is good practice! In real life problems are usually complicated, and no artificially ‘nice’ answers.