1. Convert the following diff. eq. to a system of first-order diff. eqs

\[ y^{(4)}(t) = \sqrt{2y'''(t) + t^2 + y'(t)y(t) + \cos y(t) + ey''(t)} \]

**Sol. to 1:** We define

\[ x_1(t) = y(t) \quad x_2(t) = y'(t) \quad x_3(t) = y''(t) \quad x_4(t) = y'''(t) \]

The first three equations are easy! (and always the same!)

\[ x_1'(t) = x_2(t) \]
\[ x_2'(t) = x_3(t) \]
\[ x_3'(t) = x_4(t) \]

The last equation is

\[ x_4'(t) = \sqrt{2x_4(t) + t^2 + x_2(t)x_1(t) + \cos x_1(t) + e^{x_3(t)}} \]

(Obtained by replacing \( y(t) \) by \( x_1(t) \), \( y'(t) \) by \( x_2(t) \), \( y''(t) \) by \( x_3(t) \), \( y'''(t) \) by \( x_4(t) \)).

2. Solve the initial value problem for the system, by using techniques for solving a single diff. eq. of higher order

\[ x_1'(t) = 2x_2(t) \quad x_2'(t) = -8x_1(t) \quad x_1(0) = 2 \quad x_2(0) = 32 \]

**Sol. of 2:** From the first equation we have

\[ x_2(t) = \frac{1}{2} x_1'(t) \]

Plugging into the second

\[ \left( \frac{1}{2} x_1'(t) \right)' = -8x_1(t) \]

So

\[ x_1''(t) = -16x_1(t) \]

and moving everything to the left:

\[ x_1''(t) + 16x_1(t) = 0 \]
Let’s solve this diff. eq. The characteristic equation is
\[ r^2 + 16 = 0 \]
So
\[ r^2 = -16 \]
and \( r = 0 \pm 4i \). So the general solution is
\[ x_1(t) = c_1 \cos 4t + c_2 \sin 4t \]
We need initial conditions. Of course, \( x_1(0) = 2 \), but what is \( x'_1(0) \)? Going back to:
\[ x_2(t) = \frac{1}{2} x'_1(t) \]
and plugging-in \( t = 0 \):
\[ x_2(0) = \frac{1}{2} x'_1(0) \]
So \( x'_1(0) = 2x_2(0) = 2 \cdot 32 = 64 \).
Plugging-in \( t = 0 \) into the general solution gives:
\[ x_1(0) = c_1 \cos 0 + c_2 0 = c_1 \]
So \( c_1 = 2 \). Differentiating the general solution we have
\[ x'_1(t) = -4c_1 \sin 4t + 4c_2 \cos 4t \]
So, when \( t = 0 \):
\[ x'_1(0) = 4c_2 \]
since \( x'_1(0) = 64 \) we get
\[ 64 = 4c_2 \]
so \( c_2 = 16 \).
Going back to the general solution, we have
\[ x_1(t) = 2 \cos 4t + 16 \sin 4t \]
Finally, going back to
\[ x_2(t) = \frac{1}{2} x'_1(t) \]
We have
\[ x_2(t) = \frac{1}{2} (2 \cos 4t + 16 \sin 4t)' = \frac{1}{2} (-8 \sin 4t + 64 \cos 4t) = -4 \sin 4t + 32 \cos 4t \]
**Ans. to 2:**
\[ x_1(t) = 2 \cos 4t + 16 \sin 4t \quad , \quad x_2(t) = -4 \sin 4t + 32 \cos 4t \]
**Comments:** Many people ran out of time (it was a LONG problem), but about \( 20\% \) finished it correctly! Congratulations!