1. Find the eigenvalues, and corresponding eigenvectors of the following matrix.

\[
\begin{pmatrix}
-5 & 2 \\
-6 & 2
\end{pmatrix}
\]

1. The characteristic equation is

\[
\det \begin{pmatrix}
\lambda + 5 & -2 \\
6 & \lambda - 2
\end{pmatrix} = 0.
\]

So

\[(\lambda + 5)(\lambda - 2) - (-2)(6) = \lambda^2 + 3\lambda - 10 + 12 = \lambda^2 + 3\lambda + 2 = 0\]

Factorizing

\[(\lambda + 1)(\lambda + 2) = 0,\]

so the eigenvalues are \(\lambda = -2\) and \(\lambda = -1\).

Now it is time to find the eigenvectors.

For \(\lambda = -2\) we solve

\[
\begin{pmatrix}
-5 & 2 \\
-6 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
-2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}.
\]

Spelling it out

\[-5x_1 + 2x_2 = -2x_1 \\
-6x_1 + 2x_2 = -2x_2.
\]

From the first equation we get \(3x_1 = 2x_2\), so \(x_1 = \frac{2}{3}x_2\). Plugging into the second equation gives us \(0 = 0\) and that’s good! So an eigenvector for \(\lambda = -2\) is

\[
\begin{pmatrix}
x_2 \\
x_2
\end{pmatrix} = x_2 \begin{pmatrix}
\frac{2}{3} \\
1
\end{pmatrix}.
\]

We can choose \(x_2 = 1\), but we can pick any non-zero value, and \(x_2 = 3\) is nicer (to clear fractions), so the best looking eigenvector for \(\lambda = -2\) is

\[
\begin{pmatrix}
2 \\
3
\end{pmatrix}.
\]

For \(\lambda = -1\) we solve

\[
\begin{pmatrix}
-5 & 2 \\
-6 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
-1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}.
\]

Spelling it out

\[-5x_1 + 2x_2 = -x_1.
\]
\[-6x_1 + 2x_2 = -2x_2 \ .\]

From the first equation we get \(4x_1 = 2x_2\), so \(x_2 = 2x_1\). Plugging into the second equation gives us \(0 = 0\) and that's good! So an eigenvector for \(\lambda = -1\) is

\[
\begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} .
\]

We can choose \(x_1 = 1\). So an eigenvector for \(\lambda = -1\) is

\[
\begin{pmatrix} 1 \\ 2 \end{pmatrix} .
\]

**Ans. to 1:** There are two eigenvalues, \(\lambda = -2\) and \(\lambda = -1\).

An eigenvector corresponding to \(\lambda = -2\) is \(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\). (and any non-zero multiple of it is also an eigenvector, of course)

An eigenvector corresponding to \(\lambda = -1\) is \(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\) (and any non-zero multiple of it is also an eigenvector, of course)

**2.** Write the following system in matrix notation

\[
x'_1(t) = -tx_3(t) - 2tx_2(t) + \frac{1}{t+1} \\
x'_2(t) = -x_3(t) - 3x_1(t) + \cos^2 t \\
x'_3(t) = -9x_2(t) + \tan tx_1(t) + 3t^2x_3(t)
\]

**Sol. to 2:** First we have to write \(x_1(t), x_2(t), x_3(t)\) in order, and stick 0 in front of those that are missing

\[
x'_1(t) = 0x_1(t) - 2tx_2(t) - tx_3(t) + \frac{1}{t+1} \\
x'_2(t) = -3x_1(t) + 0x_2(t) - x_3(t) + \cos^2 t \\
x'_3(t) = \tan tx_1(t) - 9x_2(t) + 3t^2x_3(t)
\]

Finally:

**Ans. to 2:**

\[
x'(t) = \begin{pmatrix} 0 & -2t & -t \\ -3 & 0 & -1 \\ \tan t & -9 & 3t^2 \end{pmatrix} x(t) + \begin{pmatrix} \frac{1}{t+1} \\ \cos^2 t \\ 0 \end{pmatrix} .
\]