Solutions to Attendance Quiz # 19 for Dr. Z.'s Calc4 for Nov. 11, 2013

1. Solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix} \mathbf{x}(t) \quad , \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad .$$

Sol. to 1: We first form the **characteristic matrix**:

$$\begin{pmatrix} -5-r & 2 \\ -6 & 2-r \end{pmatrix}$$

We next take the **determinant**

$$\det \begin{pmatrix} -5-r & 2 \\ -6 & 2-r \end{pmatrix} = (-5-r)(2-r) - 2(-6) = (r+5)(r-2) + 12 = r^2 + 3r + 2 \quad .$$

We next set it equal to 0, getting the **characteristic equation**

$$r^2 + 3r + 2 = 0 \quad .$$

We next **solve it**. Since $r^2 + 3r + 2 = (r+1)(r+2)$ we have to solve

$$(r+1)(r+2) = 0 \quad ,$$

getting two **eigenvalues**, $r = -1$ and $r = -2$.

We next, **one at a time**, find a corresponding **eigenfunction**.

When $r = -1$ the characteristic matrix is

$$\begin{pmatrix} -5-(-1) & 2 \\ -6 & 2-(-1) \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}$$

A tentavive eigenvector $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ satisfies

$$\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling it out, we have to solve the **system**

$$-4c_1 + 2c_2 = 0 \quad ,$$

$$-6c_1 + 3c_2 = 0 \quad .$$

From the first equation, we get $2c_2 = 4c_1$, so $c_2 = 2c_1$, plugging this into the second equation gives $0 = 0$ (as it should!, the second equation is not allowed to add any more information). So the **eigenvector** corresponding to $r = -1$ is

$$\begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad .$$

Putting $c_1 = 1$, we get that an eigenvector corresponding to $r = -1$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Note: Putting $c_1 = 1$ is convenient, but you may pick any **non-zero** value of c_1 above.

When $r = -2$ the characteristic matrix is

$$\begin{pmatrix} -5-(-2) & 2 \\ -6 & 2-(-2) \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}$$

A tentavive eigenvector $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ satisfies

$$\begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling it out, we have to solve the **system**

$$-3c_1 + 2c_2 = 0 \quad ,$$

$$-6c_1 + 4c_2 = 0 \quad .$$

From the first equation, we get $c_2 = \frac{3}{2}c_1$, plugging this into the second equation gives $0 = 0$ (as it should!, the second equation is not allowed to add any more information). So the **eigenvector** corresponding to $r = -2$ is

$$\begin{pmatrix} c_1 \\ \frac{3}{2}c_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \quad .$$

Putting $c_1 = 2$, we get that an eigenvector corresponding to $r = -2$ is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Note: Putting $c_1 = 2$ is convenient (to clear fractions), but you may pick any **non-zero** value of c_1 above.

Summarizing the Linear Algebra part:

$r = -1$ is an eignevalue with a corresponding eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

$r = -2$ is an eignevalue with a corresponding eigenvector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Hence the **General Solution** is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} \quad .$$

Note: These c_1, c_2 are **arbitrary constants** (no relation to the c_1, c_2 above for finding the eigenvectors).

Now it is time to use the **initial condition** $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$.

Plugging-in $t = 0$ in the general solution, we have

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad .$$

So

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad .$$

Spelling it out, we have to solve the system

$$c_1 + 2c_2 = 4 \quad ,$$

$$2c_1 + 3c_2 = 7 \quad .$$

From the first equation, we get $c_1 = 4 - 2c_2$. Putting into the second:

$$2(4 - 2c_2) + 3c_2 = 7 \quad .$$

So

$$8 - 4c_2 + 3c_2 = 7$$

$$c_2 = 1$$

and hence $c_1 = 4 - 2 \cdot 1 = 4 - 2 = 2$.

Going back to the **general solution** above, and replacing c_1 by 2 and c_2 by 1 gives

$$\mathbf{x}(t) = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} \quad .$$

This is **correct** but not **final**. The final answer is obtained by by replacing $2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ by $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, so

Ans. to 1:

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} \quad .$$