Solutions to the Attendance Quiz # 20 for Dr. Z.'s Calc4 for Lecture 20

1. Solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} -4 & 5 \\ -1 & 0 \end{pmatrix} \mathbf{x}(t) \quad ; \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad .$$

Sol. 1: The **characteristic matrix** is

$$\begin{pmatrix} -4-r & 5 \\ -1 & 0-r \end{pmatrix} = \begin{pmatrix} -4-r & 5 \\ -1 & -r \end{pmatrix} \quad .$$

The **determinant**, is

$$(-4-r)(-r) - 5(-1) = (r+4)(r) + 5 = r^2 + 4r + 5 \quad .$$

The **characteristic equation is**

$$r^2 + 4r + 5 = 0 \quad .$$

The **eigenvalues** are

$$\frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2 \cdot 1} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i \quad .$$

Whenever we have **complex** roots, it is enough to only consider **one** eigenvalue.

Let's take $r = -2 + i$.

Going back to the characteristic matrix:

$$\begin{pmatrix} -4-(-2+i) & 5 \\ -1 & -(-2+i) \end{pmatrix} = \begin{pmatrix} -2-i & 5 \\ -1 & 2-i) \end{pmatrix}$$

Next we find an **eigenvector** $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, such that

$$\begin{pmatrix} -2-i & 5 \\ -1 & 2-i) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling out, we have to solve the system

$$(-2-i)c_1 + 5c_2 = 0 \quad ,$$

$$-c_1 + (2-i)c_2 = 0 \quad .$$

We can take **either** equation, so let's take the second that is simpler. We get

$$c_1 = (2-i)c_2 \quad .$$

Puting this inside the tentative eigenvector $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (2-i)c_2 \\ c_2 \end{pmatrix} = c_2 \begin{pmatrix} 2-i \\ 1 \end{pmatrix} \quad .$$

We can take c_2 to be **anything** EXCEPT ZERO! So let's take $c_2 = 1$, and we got

$-2+i$ is an eignevalue, that has an eigenvector $\begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

So a **solution** (in the complex world!) is

$$e^{(-2+i)t} \begin{pmatrix} 2-i \\ 1 \end{pmatrix} = e^{-2t}e^{it} \begin{pmatrix} (2-i) \\ 1 \end{pmatrix} = e^{-2t} \begin{pmatrix} (2-i)e^{it} \\ e^{it} \end{pmatrix} \quad .$$

Now is time to use **Euler's Lovely Formula**

$$e^{it} = \cos t + i \sin t \quad .$$

So the above solution is:

$$e^{-2t} \begin{pmatrix} (2-i)(\cos t + i \sin t) \\ \cos t + i \sin t \end{pmatrix} \quad .$$

Expanding the top entry, we get

$$e^{-2t} \begin{pmatrix} 2\cos t + 2i \sin t - i \cos t - i^2 \sin t \\ \cos t + i \sin t \end{pmatrix} = e^{-2t} \begin{pmatrix} 2\cos t + 2i \sin t - i \cos t - (-1) \sin t \\ \cos t + i \sin t \end{pmatrix} \quad .$$

$$= e^{-2t} \begin{pmatrix} 2\cos t + 2i \sin t - i \cos t + \sin t \\ \cos t + i \sin t \end{pmatrix} = e^{-2t} \begin{pmatrix} (2\cos t + \sin t) + i(2 \sin t - \cos t) \\ \cos t + i \sin t \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t \end{pmatrix} + ie^{-2t} \begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix} \quad .$$

Both **real part** $e^{-2t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t \end{pmatrix}$ and **imaginary part** $e^{-2t} \begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix}$ are **solutions**, and they are **independent**, so the **general solution is**

$$\mathbf{x}(t) = e^{-2t} \left(c_1 \begin{pmatrix} 2\cos t + \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix} \right) \quad .$$

Note: these c_1, c_2 have no relation to the previous c_1, c_2 to find the eigenvector.

Now it is time to find c_1, c_2 by taking advantage of the **initial condition** $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Plugging-in $t = 0$ in the general solution gives

$$\mathbf{x}(0) = e^{-2\cdot 0}\left(c_1\begin{pmatrix}(2\cos 0 + \sin 0)\\ \cos 0\end{pmatrix} + c_2\begin{pmatrix}2\sin 0 - \cos 0\\ \sin 0\end{pmatrix}\right) \quad .$$

$$c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}2c_1 - c_2\\ c_1\end{pmatrix} \quad .$$

Since $\mathbf{x}(0) = \begin{pmatrix}1\\-2\end{pmatrix}$, we have

$$\begin{pmatrix}2c_1 - c_2\\ c_1\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix} \quad .$$

Spelling it out, we get a system

$$2c_1 - c_2 = 1$$

$$c_1 = -2 \quad .$$

From the second equation we have $c_1 = -2$. Plugging into the first equation, we have

$$2(-2) - c_2 = 1$$

$$-4 - c_2 = 1 \quad ,$$

so $c_2 = -5$.

Plugging-in $c_1 = -2$, $c_2 = -5$ into the general solution gives

$$\mathbf{x}(t) = e^{-2t}\left(-2\begin{pmatrix}2\cos t + \sin t\\ \cos t\end{pmatrix} - 5\begin{pmatrix}2\sin t - \cos t\\ \sin t\end{pmatrix}\right) \quad .$$

This is **perfectly correct**, but it can be simplified.

$$\mathbf{x}(t) = e^{-2t}\left(\begin{pmatrix}-4\cos t - 2\sin t\\ -2\cos t\end{pmatrix} + \begin{pmatrix}-10\sin t + 5\cos t\\ -5\sin t\end{pmatrix}\right)$$

$$= e^{-2t}\begin{pmatrix}\cos t - 12\sin t\\ -2\cos t - 5\sin t\end{pmatrix} \quad .$$

Ans. to 1:

$$\mathbf{x}(t) = e^{-2t}\begin{pmatrix}\cos t - 12\sin t\\ -2\cos t - 5\sin t\end{pmatrix} \quad .$$