1.

(i) Seek power series solution of the given differential equation at \( x_0 = 0 \), find the recurrence relation.

(ii) Find the first four terms in each of two solutions \( y_1(x), y_2(x) \) (unless the series terminates sooner)

\[ y''(x) - xy'(x) - y(x) = 0 \]

Sol. to 1(i): We first write the infinite template

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]

We next find expressions, as infinite series, for \( y'(x) \) and \( y''(x) \):

\[ y'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} \]

\[ y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \]

We next plug everything into the diff.eq.

\[ \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=0}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \]

We next bring the \( x \) in front of the middle \( \sum \) inside, getting

\[ \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \]

But the first two terms of the first \( \sum \) are zero, i.e. when \( n = 0 \) and \( n = 1 \) it contributes nothing, so we can start the \( \sum \) at \( n = 2 \) getting

\[ \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \]

We next make a change of discrete variable in the first \( \sum \), putting \( m = n - 2 \) and hence \( n = m + 2 \). The sum becomes an \( m \)-sum starting at \( m = 0 \),

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m + 2)(m + 1)a_{m+2} x^m \]
But $m$ is just a ‘dummy variable’, so we can go back to using $n$ as summation index, in order to be consistent with the other $\sum$’s. Going back above we have

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0 .$$

Collecting terms

$$\sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} - (n+1)a_n)x^n = 0 .$$

In order for the series on the left side to be **identically zero**, **EVERY COEFFICIENT MUST BE ZERO**. So we set the coefficient of $x^n$ to 0, getting a recurrence

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0 .$$

Dividing by $n+1$ and moving $a_n$ to the right side, we get:

$$a_{n+2} = \frac{1}{n+2} a_n .$$

**Ans. to 1(i)**: The recurrence for the coefficients is $a_{n+2} = \frac{1}{n+2} a_n$.

**Sol. to part(ii)*** We use the recurrence, one step at a time.

When $n = 0$ we get

$$a_2 = \frac{1}{2} a_0 .$$

When $n = 1$,

$$a_3 = \frac{1}{3} a_1 .$$

When $n = 2$,

$$a_4 = \frac{1}{4} a_2 .$$

But we already know what $a_2$ is, so

$$a_4 = \frac{1}{2 \cdot 4} a_0 .$$

When $n = 3$,

$$a_5 = \frac{1}{5} a_3 .$$

But we already know what $a_3$ is, so

$$a_4 = \frac{1}{5 \cdot 3} a_1 .$$

When $n = 4$,

$$a_6 = \frac{1}{6} a_4 .$$
But we already know what $a_4$ is, so
\[ a_6 = \frac{1}{6 \cdot 8} a_0 . \]
When $n = 5$,
\[ a_7 = \frac{1}{7} a_5 . \]
But we already know what $a_5$ is, so
\[ a_7 = \frac{1}{7 \cdot 5 \cdot 3} a_1 . \]

Putting it into $y(x)$, we get
\[ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \ldots \]
\[ a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{2 \cdot 4} a_0 x^4 + \frac{1}{3 \cdot 5} a_1 x^5 + \frac{1}{2 \cdot 4 \cdot 6} a_0 x^6 + \frac{1}{3 \cdot 5 \cdot 7} a_1 x^7 + \ldots \]

We now separate the $a_0$ and $a_1$ terms, getting
\[ y(x) = a_0 \left( 1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1}{2 \cdot 4 \cdot 6} x^6 + \ldots \right) , \]
\[ + a_1 \left( x + \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \frac{1}{3 \cdot 5 \cdot 7} x^7 + \ldots \right) . \]

**Ans. to 1(ii):**

The first four terms of the fundamental solutions $y_1(x)$ and $y_2(x)$ are
\[ y_1(x) = 1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1}{2 \cdot 4 \cdot 6} x^6 + \ldots , \]
\[ y_2(x) = x + \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \frac{1}{3 \cdot 5 \cdot 7} x^7 + \ldots . \]