Important Definitions: Discretization

The discrete approximations of the second derivatives with mesh-size $h$ are:

\[ u_{xx} \approx \frac{1}{h^2} [u(x + h, y) - 2u(x, y) + u(x - h, y)] , \]
\[ u_{yy} \approx \frac{1}{h^2} [u(x, y + h) - 2u(x, y) + u(x, y - h)] . \]

The five-point approximation of the Laplacian $u_{xx} + u_{yy}$ (in 2D) is

\[ u_{xx} + u_{yy} \approx \frac{1}{h^2} [u(x + h, y) + u(x, y + h) + u(x - h, y) + u(x, y - h) - 4u(x, y)] \]

To numerically (approximately) solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ in a region $D$ with boundary condition $u(x, y) = F(x, y)$ along the boundary with mesh-size $h$, you set $u_{i,j} = u(ih, jh)$ and set up a system of linear equation as follows.

For each $(ih, jh)$ inside the region, you have an equation

\[ u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = 0 , \]

and for every boundary point

\[ u_{i,j} = F(ih, jh) . \]

Then do the linear algebra, and the solutions, $\{u_{i,j}\}$ would give you approximations for the values of the “real thing” at the interior points $\{(ih, jh)\}$.

**Problem 0.1**: Approximate, with mesh-size $h = 1$, the solution of the boundary-value problem

\[ u_{xx} + u_{yy} = 0 , \quad 0 < x < 3 , \quad 0 < y < 3 ; \]

subject to the boundary conditions

\[ u(0, y) = 2y , \quad 0 < y < 3 \quad ; \quad u(3, y) = -y , \quad 0 < y < 3 \; ; \]
\[ u(x, 0) = -x , \quad 0 < x < 3 \quad ; \quad u(x, 3) = 2x , \quad 0 < x < 3 \; . \]

**Solution**: There are 12 points. 8 are on the boundary, and four are inside. The boundary points are

\[ P_{10} = (1, 0) , \quad P_{20} = (2, 0) , \]
\[ P_{13} = (1, 3) , \quad P_{23} = (2, 3) , \]
\[ P_{01} = (0, 1) \quad , \quad P_{02} = (0, 2) \quad , \]
\[ P_{31} = (3, 1) \quad , \quad P_{32} = (3, 2) \quad . \]

\( P_{10} \) and \( P_{20} \) are on \( y = 0 \), so using the data \( u(x, 0) = -x \), \( 0 < x < 3 \), we get
\[ u_{10} = u(P_{10}) = u(1, 0) = -1 \quad , \quad u_{20} = u(P_{20}) = u(2, 0) = -2 \quad . \]

\( P_{13} \) and \( P_{23} \) are on \( y = 3 \), so using the data \( u(x, 3) = 2x \), \( 0 < x < 3 \), we get
\[ u_{13} = u(P_{13}) = u(1, 3) = 2 \cdot 1 = 2 \quad , \quad u_{23} = u(P_{23}) = u(2, 3) = 2 \cdot 2 = 4 \quad . \]

\( P_{01} \) and \( P_{02} \) are on \( x = 0 \), so using the data \( u(0, y) = 2y \), \( 0 < y < 3 \), we get
\[ u_{01} = u(P_{01}) = u(0, 1) = 2 \cdot 1 = 2 \quad , \quad u_{02} = u(P_{02}) = u(0, 2) = 2 \cdot 2 = 4 \quad . \]

\( P_{31} \) and \( P_{32} \) are on \( x = 3 \), so using the data \( u(3, y) = -y \), \( 0 < y < 3 \), we get
\[ u_{31} = u(P_{31}) = u(3, 1) = -1 \quad , \quad u_{32} = u(P_{32}) = u(3, 2) = -2 \quad . \]

Summarizing, we have the following data regarding the boundary
\[ u_{10} = -1 \quad , \quad u_{20} = -2 \quad , \quad u_{13} = 2 \quad , \quad u_{23} = 4 \quad , \]
\[ u_{01} = 2 \quad , \quad u_{02} = 4 \quad , \quad u_{31} = -1 \quad , \quad u_{32} = -2 \quad . \]

Regarding the interior points we have.

Point (1,1):
\[ u_{1+1,1} + u_{1,1+1} + u_{1-1,1} + u_{1,1-1} - 4u_{1,1} = 0 \quad , \]
meaning
\[ u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0} - 4u_{1,1} = 0 \quad . \]

Point (2,1):
\[ u_{2+1,1} + u_{2,1+1} + u_{2-1,1} + u_{2,1-1} - 4u_{2,1} = 0 \quad , \]
meaning
\[ u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0} - 4u_{2,1} = 0 \quad . \]

Point (1,2):
\[ u_{1+1,2} + u_{1,2+1} + u_{1-1,2} + u_{1,2-1} - 4u_{1,2} = 0 \quad , \]
meaning \[ u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1} - 4u_{1,2} = 0 \ . \]

Point (2, 2):

\[ u_{2+1,2} + u_{2,2+1} + u_{2-1,2} + u_{2,2-1} - 4u_{2,2} = 0 \ , \]

meaning \[ u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1} - 4u_{2,2} = 0 \ . \]

So we have the following four linear equations:

\[ u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0} - 4u_{1,1} = 0 \ , \]
\[ u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0} - 4u_{2,1} = 0 \ , \]
\[ u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1} - 4u_{1,2} = 0 \ , \]
\[ u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1} - 4u_{2,2} = 0 \ . \]

Now we have to plug-in the known values for the boundary points, namely:

\[ u_{10} = -1 \ , \ u_{20} = -2 \ , \ u_{13} = 2 \ , \ u_{23} = 4 \ , \]
\[ u_{01} = 2 \ , \ u_{02} = 4 \ , \ u_{31} = -1 \ , \ u_{32} = -2 \ . \]

Our system of linear equations for the unknowns \( u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2} \) becomes:

\[ u_{2,1} + u_{1,2} + 2 + (-1) - 4u_{1,1} = 0 \ , \]
\[ (-1) + u_{2,2} + u_{1,1} + (-2) - 4u_{2,1} = 0 \ , \]
\[ u_{2,2} + (2) + (4) + u_{1,1} - 4u_{1,2} = 0 \ , \]
\[ -2 + 4 + u_{1,2} + u_{2,1} - 4u_{2,2} = 0 \ . \]

Moving all numbers to the right side, and writing each equation in the order \( u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2} \), we get the system:

\[ -4u_{1,1} + u_{2,1} + u_{1,2} = -1 \ , \]
\[ u_{1,1} - 4u_{2,1} + u_{2,2} = 3 \ , \]
\[ u_{1,1} - 4u_{1,2} + u_{2,2} = -6 \ , \]
\[ u_{2,1} + u_{1,2} - 4u_{2,2} = -2 \ . \]
Or in matrix notation
\[
\begin{pmatrix}
-4 & 1 & 1 & 0 \\
1 & -4 & 0 & 1 \\
1 & 0 & -4 & 1 \\
0 & 1 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
u_{1,1} \\
u_{2,1} \\
u_{1,2} \\
u_{2,2}
\end{pmatrix}
=
\begin{pmatrix}
-1 \\
3 \\
-6 \\
-2
\end{pmatrix}.
\]
Solving this system, either by hand, using Gaussian elimination, or using Matlab or Maple, we get the solutions:
\[
u_{11} = \frac{5}{8}, \quad u_{21} = -\frac{3}{8}, \quad u_{12} = \frac{15}{8}, \quad u_{22} = \frac{7}{8}.
\]
In other words, the approximations for the interior points are

**Ans. to 0.1:**
\[
u(1,1) \approx \frac{5}{8}, \quad u(2,1) \approx -\frac{3}{8}, \quad u(1,2) \approx \frac{15}{8}, \quad u(2,2) \approx \frac{7}{8}.
\]