The Three Most Important PDEs

Heat Equation: (Parabolic Type)

\[ k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0. \]

(where \( k \) is a positive number).

Wave Equation (Hyperbolic Type)

\[ \alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0. \]

Laplace’s Equation (Elliptic Type)

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b. \]

Types of Initial Conditions

For the Heat Equation: \( u(x,0) = f(x) \), where \( f(x) \) is the temperature at point \( x \) on the rod \([0,L]\) at the beginning \((t = 0)\).

For the Wave Equation: \( u(x,0) = f(x) \), where \( f(x) \) is how the stretched string looks at the beginning \((t = 0)\), and \( u_t(x,0) = g(x) \), where \( g(x) \) is its velocity function at the very beginning.

For Laplace’s Equation: None, \( u(x,y) \) is a function of \((x,y)\) not of time, \( t \).

Types of Boundary Conditions:

For the Heat Equation:

\( u(0,t) = u_0 \), if the left-side is kept at constant temperature \( u_0 \).

\( u(L,t) = u_0 \), if the right-side is kept at constant temperature \( u_0 \).

\( u_x(0,t) = 0 \), if the left-side is insulated.

\( u_x(L,t) = 0 \), if the right-side is insulated.

Wave Equation:

\( u(0,t) = 0, \ u(L,t) = 0, \ t > 0, \) if both ends of the strings are secured to the \( x \)-axis.
u(0, t) = f(t), t > 0 if the left side has transversal motion of the form \( f(t) \).

\( u(L, t) = g(t), t > 0 \) if the right side has transversal motion of the form \( g(t) \).

**Laplace’s Equation:** If \( u(x, y) \) is defined in \( 0 < x < a, 0 < y < b \), then the following types may show up (but not all of them at once)

\[
\begin{align*}
\quad u(0, y) &= f(y), \quad 0 < y < b \quad \text{(left side held at temperature \( f(y) \))} \\
\quad u(a, y) &= g(y), \quad 0 < y < b \quad \text{(right side held at temperature \( g(y) \))} \\
\quad u(x, 0) &= F(x), \quad 0 < x < a \quad \text{(bottom side held at temperature \( F(x) \))} \\
\quad u(x, b) &= G(x), \quad 0 < x < a \quad \text{(top side held at temperature \( G(x) \))}
\end{align*}
\]

\[
\begin{align*}
\quad u_x(0, y) &= 0, \quad 0 < y < b \quad \text{(left side is insulated)} \\
\quad u_x(a, y) &= 0, \quad 0 < y < b \quad \text{(right side is insulated)} \\
\quad u_y(x, 0) &= 0, \quad 0 < x < a \quad \text{(bottom side is insulated)} \\
\quad u_y(x, b) &= 0, \quad 0 < x < a \quad \text{(top side is insulated)}
\end{align*}
\]

**Problem 14.1:** A rod of length \( L \) coincides with the interval \([0, L]\) on the \( x \)-axis. Set up the boundary value problem for the temperature \( u(x, t) \).

- **a.** The left end is insulated, the right-end is held at temperature 100, the initial temperature is \( f(x) \).
- **b.** The left end is held at temperature \( u_0 \), the right end is insulated and initial temperature is 0 throughout.

**Solution:** The pde for temperature of rods \( u(x, t) \) is always

\[
k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0, 0 < x < L, \quad t > 0.
\]

Only the initial and boundary conditions change from one problem to the next.

For **a.** \( u_x(0, t) = 0 \) (since the left end is insulated), \( u(L, t) = 100, u(x, 0) = f(x) \).

For **b.** \( u(0, t) = u_0 \) (since the left end is always at temperature \( u_0 \)), \( u_x(L, t) = 0 \) (since the right end is insulated), \( u(x, 0) = 0 \).

**Problem 14.2:** A string of length \( L \) coincides with the interval \([0, L]\) on the \( x \)-axis. Set up the boundary-value problem for the displacement \( u(x, t) \).
a. The ends are secured to the $x$-axis. The string is released from rest from the initial displacement $x^2(L - x)^2$.

b. The ends are secured to the $x$-axis. The string is along the $x$-axis at the very beginning, but has initial velocity $\sin(\pi x / L)$.

c. The right end is secured to the $x$-axis, but the left end moves in a transversal manner according to $\sin(3\pi t)$. Initially the string is undisplaced and is at rest.

**Solution.** For displacement of a string the pde is always the Wave Equation:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0.$$ 

Only the initial and boundary conditions change.

For a: $u(0,t) = 0, u(L,t) = 0$ (since both ends are secured); $u(x,0) = x^2(L - x)^2$ (that’s how it looks at the start), $u_t(x,0) = 0$ (since it starts at rest).

For b: $u(0,t) = 0, u(L,t) = 0$ (since both ends are secured); $u(x,0) = 0, u_t(x,0) = \sin(\pi x / L)$,

For c: $u(0,t) = \sin(3\pi t)$ (that’s how it moves at $x = 0$) $u(L,t) = 0$ (the right end is secured), $u(x,0) = 0$ (since initially it is undisplaced), $u_t(x,0) = 0$ (since initially it is at rest).

**Problem 14.3:** Set up the boundary value problem for the steady-state temperature $u(x,y)$, where a thin rectangular plate coincides with the region in the $xy$-plane defined by $0 \leq x \leq 10, 0 \leq y \leq 20$. The left end and the bottom of the plate are insulated, the top of the plate is held at temperature 50, and the right end of the plate is held at temperature $g(y)$.

**Solution:** The pde for “steady-state temperature” (in the plane) is always Laplace’s Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$$

for the appropriate $a$ and $b$. In this problem $a = 10, b = 20$, so the pde is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 10, \quad 0 < y < 20.$$

Since the left end ($x = 0, 0 < y < 20$) is insulated we have: $u_x(0,y) = 0$.

Since the bottom end ($0 < x < 10, y = 0$) is insulated we have: $u_y(x,0) = 0$.

Since the top of the plate ($0 < x < 10, y = 20$), is held at temperature 50, we have $u(x,20) = 50$.

Since the right end of the plate ($x = 10, 0 < y < 20$), is held at temperature $g(y)$, we have $u(10,y) = g(y)$.

**Answer to 14.3:**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 10, \quad 0 < y < 20,$$
subject to the boundary conditions

\[ u_x(0,y) = 0 , \quad 0 < y < 20 \ ; \quad u(10,y) = g(y) , \quad 0 < y < 20 \ ; \]

\[ u_y(x,0) = 0 , \quad 0 < x < 10 \ ; \quad u(x,20) = 50 , \quad 0 < x < 10 \ . \]