Theory: The Laplace Transform is a dictionary that goes from functions of $t$ (usually time) to functions of $s$. It is often necessary to be able to translate back! Taking the little table of Lecture 1, and reversing it we get:

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

(b) $\mathcal{L}^{-1}\left\{\frac{1}{s^k}\right\} = \frac{t^{k-1}}{(k-1)!} (k = 1, 2, 3, ...)$,

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$,

(d) $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\} = \frac{\sin kt}{k}$,

(e) $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\} = \cos kt$,

(f) $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - k^2}\right\} = \frac{\sinh kt}{k}$,

(g) $\mathcal{L}^{-1}\left\{\frac{s}{s^2 - k^2}\right\} = \cosh kt$.

Note that except for (a), each of these formulas contains infinitely many facts, since they involve parameters. For example thanks to (b) we know the inverse-Laplace-Transform of $1/s, 1/s^2, 1/s^3$ etc.

Problem 2.1: Find $\mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5}{s-5}\right\}$

Solution: By linearity:

$$\mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5}{s-5}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} .$$

By (a) (or (b) with $k = 1$)

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 .$$

By (c)

$$\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = e^{5t} .$$

Combining, we have

$$\mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5}{s-5}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = 3 \cdot 1 + 5e^{5t} = 3 + 5e^{5t} .$$

Ans. to 2.1: $\mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5}{s-5}\right\} = 3 + 5e^{5t} .
Problem 2.2: Find $\mathcal{L}^{-1}\left\{ \frac{2s+1}{s^2+9} \right\}$

**Sol.:** Remember, we can always break-up the numerator!

\[
\mathcal{L}^{-1}\left\{ \frac{2s+1}{s^2+9} \right\} = \mathcal{L}^{-1}\left\{ \frac{2s}{s^2+9} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{s^2+9} \right\} = 2\mathcal{L}^{-1}\left\{ \frac{s}{s^2+9} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{s^2+9} \right\} .
\]

By (e) and (d) with $k = 3$, this equals

$$2 \cos 3t + \frac{1}{3} \sin 3t .$$

**Ans. to 2.2:** $\mathcal{L}^{-1}\left\{ \frac{2s+1}{s^2+9} \right\} = 2 \cos 3t + \frac{1}{3} \sin 3t.$

When we get **complicated** rational functions of $s$, we need to do a **partial fraction** decomposition.

**Problem 2.3:** Evaluate

$$\mathcal{L}^{-1}\left\{ \frac{3s-4}{s^2-3s+2} \right\}$$

**Sol.:** First we must **factorize** the denominator (unless it is already factored)

$$\frac{3s-4}{s^2-3s+2} = \frac{3s-4}{(s-1)(s-2)} .$$

Now are are looking for **magic** numbers, let’s call them $A$ and $B$, such that

$$\frac{3s-4}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} .$$

Adding the fractions on the right (using $a/b + c/d = (ad+bc)/bd$):

$$\frac{3s-4}{(s-1)(s-2)} = \frac{A(s-2)+B(s-1)}{(s-1)(s-2)} .$$

The denominators automatically match, but to make this come true, we need the numerators to match:

$$3s-4 = A(s-2) + B(s-1) .$$

Now we plug-in **convenient values.** When $s = 2$ we get

$$3 \cdot 2 - 4 = A(2-2) + B(2-1) = 0 + B = B .$$

So $B = 2$. When $s = 1$ we get:

$$3 \cdot 1 - 4 = A(1-2) + B(1-1) .$$

So

$$-1 = -A$$

$$2$$
and so $A = 1$. Once we know what $A$ and $B$ are we go back and write:

$$\frac{3s - 4}{(s-1)(s-2)} = \frac{1}{s-1} + \frac{2}{s-2}.$$ 

So far it was just algebra. Only now are we ready to apply $L^{-1}$.

$$L^{-1}\left\{ \frac{3s - 4}{s^2 - 3s + 2} \right\} = L^{-1}\left\{ \frac{1}{s-1} \right\} + L^{-1}\left\{ \frac{2}{s-2} \right\} = L^{-1}\left\{ \frac{1}{s-1} \right\} + 2L^{-1}\left\{ \frac{1}{s-2} \right\}.$$ 

Using (c) with $a = 1$ and $a = 2$, we get:

$$e^t + 2e^{2t}.$$ 

Ans. to 2.3:

$$L^{-1}\left\{ \frac{3s - 4}{s^2 - 3s + 2} \right\} = e^t + 2e^{2t}.$$ 

More Theory: The beauty of $L$ is that it turns differential equations into algebraic equations!

If $y(t)$ is any function of time, and if $Y(s)$ is its Laplace Transform:

$$L\{y(t)\} = Y(s),$$

then the Laplace Transform of the first derivative, $y'(t)$, is “almost” $Y(s)$ multiplied by $s$, and if $y(0) = 0$ then it is exactly that. We have:

$$L\{y'(t)\} = sY(s) - y(0)$$

(Prove this! Hint: Integration by parts.) Applying this very same formula to $y''(t)$ we have

$$L\{y''(t)\} = s(sY(s) - y(0)) - y'(0) = s^2Y(s) - sy(0) - y'(0).$$

One more time:

$$L\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0),$$

and so on. In general:

$$L\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{(n-1)}(0).$$

Equipped with this, we can solve Initial Value Problems.

**Problem 2.4:** Use Laplace Transform to solve the following initial-value problem.

$$y' + 2y = e^t, \quad y(0) = -3.$$ 

**Solution:** Let $Y(s) = L\{y(t)\}$. Applying $L$:

$$L\{y' + 2y\} = L\{e^t\}.$$ 

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So
\[\mathcal{L}\{y\}' + 2\mathcal{L}\{y\} = \mathcal{L}\{e^t\}\ .\]

But \(\mathcal{L}\{y\}' = sY(s) - y(0) = sY(s) + 3\). Also \(\mathcal{L}\{e^t\} = \frac{1}{s-1}\). So we have the algebraic equation:

Let’s abbreviate, and write \(Y\) for \(Y(s)\)

\[sY + 3 + 2Y = \frac{1}{s-1}\ .\]

Solving for \(Y\), we get,

\[(s + 2)Y = \frac{1}{s-1} - 3 = \frac{4 - 3s}{s-1}\ .\]

Dividing both sides by \((s + 2)\), we have an explicit expression for \(Y\):

\[Y(s) = \frac{4 - 3s}{(s-1)(s+2)}\ .\]

In order to find \(y(t)\) we need to compute

\[\mathcal{L}^{-1}\{\frac{4 - 3s}{(s-1)(s+2)}\}\ .\]

Doing partial fractions:

\[Y(s) = \frac{4 - 3s}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}\ .\]

We get

\[\frac{4 - 3s}{(s-1)(s+2)} = \frac{A(s+2) + B(s-1)}{(s-1)(s+2)}\ ,\]

so

\[4 - 3s = A(s+2) + B(s-1)\ .\]

When \(s = 1\), we get \(1 = 3A\) so \(A = \frac{1}{3}\) and when \(s = -2\) we get \(10 = B(-3)\) so \(B = -\frac{10}{3}\), so the simplified expression for \(Y\) is:

\[Y(s) = \frac{\frac{1}{3}}{s-1} + \frac{-\frac{10}{3}}{s+2}\ .\]

Now we are ready to take \(\mathcal{L}^{-1}\)

\[y(t) = \mathcal{L}^{-1}\{\frac{\frac{1}{3}}{s-1} + \frac{-\frac{10}{3}}{s+2}\} = \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s-1}\} + (-\frac{10}{3})\mathcal{L}^{-1}\{\frac{1}{s+2}\} = \frac{1}{3}e^t - \frac{10}{3}e^{-2t}\ .\]

**Ans. to 2.4:** \(y(t) = \frac{1}{3}e^t - \frac{10}{3}e^{-2t}\).

**Problem 2.5:** Use Laplace Transform to solve the following initial-value problem.

\[y'' - 3y' + 2y = e^{3t}\ ,\quad y(0) = 0\ ,\quad y'(0) = 0\ .\]
Solution: As usual $\mathcal{L}\{y(t)\} = Y(s)$. Applying $\mathcal{L}$:

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{3t}\}, \quad y(0) = 0, \quad y'(0) = 0.$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s - 3}.$$ 

Note that $\mathcal{L}\{y'\} = sY - y(0) = sY$, $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y$, so

$$s^2Y - 3sY + 2Y = \frac{1}{s - 3}.$$ 

$$(s^2 - 3s + 2)Y = \frac{1}{s - 3}.$$ 

$$(s - 1)(s - 2)Y = \frac{1}{s - 3}.$$ 

Solving for $Y$:

$$Y = \frac{1}{(s - 1)(s - 2)(s - 3)}.$$ 

Partial Fractions:

$$\frac{1}{(s - 1)(s - 2)(s - 3)} = \frac{A}{s - 1} + \frac{B}{s - 2} + \frac{C}{s - 3},$$

$$1 = A(s - 2)(s - 3) + B(s - 1)(s - 3) + C(s - 1)(s - 2).$$

Convenient values: $s = 1$: $1 = A(-1)(-2)$, so $A = \frac{1}{2}$; $s = 2$: $1 = B(1)(-1)$, so $B = -1$; $s = 3$: $1 = C(2)(1)$, so $C = \frac{1}{2}$. So

$$Y = \frac{\frac{1}{2}}{s - 1} + \frac{-1}{s - 2} + \frac{\frac{1}{2}}{s - 3}.$$ 

Finally,

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{2} \frac{1}{s - 1} - \frac{1}{s - 2} + \frac{\frac{1}{2}}{s - 3}\right\}$$

$$= \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} + \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s - 3}\right\}$$

$$= \frac{1}{2} e^t - e^{2t} + \frac{1}{2} e^{3t}.$$ 

Ans. to 2.5: $y(t) = \frac{1}{2} e^t - e^{2t} + \frac{1}{2} e^{3t}$. 

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