Important Definition: Fourier Integral Representation

The Fourier Integral Representation of a function \( f(x) \) defined on the real line \((-\infty, \infty)\) is given by

\[
\frac{1}{\pi} \int_{0}^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] \, d\alpha,
\]

where

\[
A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx
\]

\[
B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx
\]

Important Theorem: If \( f(x) \) is well-behaved and \( \int_{-\infty}^{\infty} |f(x)| \, dx \) is finite, then the Fourier integral of \( f(x) \) “converges” to it (in the sense of an improper integral over \((0, \infty))\), if \( f(x) \) is continuous. If it is piece-wise continuous, it converges to it everywhere except at the discontinuities (in which case it is the average of the limits from the left and right).

Problem 20.1: Find the Fourier integral representation of the function

\[
f(x) = \begin{cases} 
0, & \text{if } x < -1; \\
1, & \text{if } -1 \leq x < 2; \\
0, & \text{if } x > 2;
\end{cases}
\]

Solution:

\[
A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx = \int_{-1}^{-1} f(x) \cos \alpha x \, dx + \int_{1}^{1} f(x) \cos \alpha x \, dx + \int_{2}^{\infty} f(x) \cos \alpha x \, dx
\]

\[
= \int_{-1}^{-1} 0 \cdot \cos \alpha x \, dx + \int_{1}^{1} 1 \cdot \cos \alpha x \, dx + \int_{2}^{\infty} 0 \cdot \cos \alpha x \, dx = 0 + \int_{-1}^{1} \cos \alpha x \, dx + 0 = \int_{-1}^{1} \cos \alpha x \, dx
\]

\[
= \sin \alpha x \bigg|_{-1}^{1} = \frac{\sin(\alpha(2)) - \sin(\alpha(-1))}{\alpha} = \frac{\sin(2\alpha) - \sin(-\alpha)}{\alpha} = \frac{\sin 2\alpha + \sin \alpha}{\alpha}.
\]

Analogously,

\[
B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx = \int_{-1}^{-1} f(x) \sin \alpha x \, dx + \int_{1}^{1} f(x) \sin \alpha x \, dx + \int_{2}^{\infty} f(x) \sin \alpha x \, dx
\]

\[
= \int_{-1}^{-1} 0 \cdot \sin \alpha x \, dx + \int_{1}^{1} 1 \cdot \sin \alpha x \, dx + \int_{2}^{\infty} 0 \cdot \sin \alpha x \, dx = 0 + \int_{-1}^{1} \sin \alpha x \, dx + 0 = \int_{-1}^{1} \sin \alpha x \, dx
\]

\[
= -\cos \alpha x \bigg|_{-1}^{1} = -\frac{\cos(\alpha(2)) - \cos(\alpha(-1))}{\alpha} = -\frac{\cos(2\alpha) - \cos(-\alpha)}{\alpha} = -\frac{\cos 2\alpha - \cos \alpha}{\alpha}.
\]
Putting both $A(\alpha)$ and $B(\alpha)$ into the formula for the Fourier Integral of $f(x)$,
\[
\frac{1}{\pi} \int_0^\infty \left[ A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x \right] d\alpha ,
\]
we get:
\[
\frac{1}{\pi} \int_0^\infty \left[ \frac{\sin 2\alpha + \sin \alpha}{\alpha} \cos \alpha x - \frac{\cos 2\alpha - \cos \alpha}{\alpha} \sin \alpha x \right] d\alpha .
\]
This is a correct answer, but using trig. identities, we can get a nicer answer:
\[
\frac{1}{\pi} \int_0^\infty \frac{\sin 2\alpha \cos \alpha x - \cos 2\alpha \sin \alpha x}{\alpha} d\alpha = \frac{1}{\pi} \int_0^\infty \frac{\sin(2\alpha - \alpha x) + \sin(\alpha + \alpha x)}{\alpha} d\alpha
\]
\[
= \frac{1}{\pi} \int_0^\infty \frac{\sin(\alpha(2 - x)) + \sin(\alpha(1 + x))}{\alpha} d\alpha
\]
Ans. to 20.1: The Fourier integral representation of $f(x)$ of the problem is \( \frac{1}{\pi} \int_0^\infty \frac{\sin(\alpha(2 - x)) + \sin(\alpha(1 + x))}{\alpha} d\alpha \).

Important Definition: Fourier Cosine Integral

The Fourier Cosine Integral of an even function $f(x)$ defined on the real line $(-\infty, \infty)$ is the cosine integral
\[
\frac{2}{\pi} \int_0^\infty A(\alpha) \cos \alpha x d\alpha ,
\]
where
\[
A(\alpha) = \int_0^\infty f(x) \cos \alpha x dx
\]

Important Definition: Fourier Sine Integral

The Fourier Integral of an odd function $f(x)$ defined on the real line $(-\infty, \infty)$ is the sine integral
\[
\frac{2}{\pi} \int_0^\infty B(\alpha) \sin \alpha x d\alpha ,
\]
where
\[
B(\alpha) = \int_0^\infty f(x) \sin \alpha x dx
\]

Note: If the function $f(x)$ is only defined on the positive real line: $(0, \infty)$, then you can extend it either as an even function, and get a Fourier cosine integral, or as an odd function, and get a Fourier sine integral. Both of them are correct.

Problem 20.2: Find the cosine and sine integral representation of the function $f(x) = xe^{-3x}, x > 0$.

Solution:
\[
A(\alpha) = \int_0^\infty xe^{-3x} \cos \alpha x dx .
\]
Using Maple, or a table of integrals (or if you have half an hour to spare, integration by parts), we have

\[ A(\alpha) = \frac{9 - \alpha^2}{(\alpha^2 + 9)^2} \].

Similarly,

\[ B(\alpha) = \int_0^\infty xe^{-3x} \sin \alpha x \, dx \]

Using Maple, or a table of integrals, we have

\[ B(\alpha) = \frac{6\alpha}{(\alpha^2 + 9)^2} \].

Putting it into the cosine and sine integrals, we have

Fourier-cosine representation:

\[ \frac{2}{\pi} \int_0^\infty \frac{(9 - \alpha^2) \cos \alpha x}{(\alpha^2 + 9)^2} \, d\alpha \]

Fourier-sine representation:

\[ \frac{2}{\pi} \int_0^\infty \frac{6\alpha}{(\alpha^2 + 9)^2} \sin \alpha x \, d\alpha = \frac{12}{\pi} \int_0^\infty \frac{\alpha}{(\alpha^2 + 9)^2} \sin \alpha x \, d\alpha \].

Ans. to 20.2: The Fourier-cosine representation of \( f(x) \) is \( \frac{2}{\pi} \int_0^\infty \frac{(9 - \alpha^2) \cos \alpha x}{(\alpha^2 + 9)^2} \, d\alpha \) and the Fourier-sine representation is \( \frac{12}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{(\alpha^2 + 9)^2} \, d\alpha \).