Dr. Z.’s Calc5 Lecture 6 Handout:
Using the Laplace Transform to solve Systems of Linear Differential Equations

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Systems of ODEs

When we have one differential equation, with one unknown function $y(t)$ and we use the Laplace Transform method, we trade it with one algebraic equation in $Y(s)$. When we have several equations with several unknown functions we trade it with a system of algebraic equations in the Laplace Transforms. We then solve the system, and at the end, transform back.

Problem 6.1: Solve the system

\[
\begin{align*}
\frac{dx}{dt} &= -x + y \\
\frac{dy}{dt} &= 2x
\end{align*}
\]

$x(0) = 0$, $y(0) = 1$.

Solution: Let $\mathcal{L}\{x\} = X(s), \mathcal{L}\{y\} = Y(s)$. Applying $\mathcal{L}$ to both equations we get

\[
\begin{align*}
\mathcal{L}\left\{\frac{dx}{dt}\right\} &= -\mathcal{L}\{x\} + \mathcal{L}\{y\} \\
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= 2\mathcal{L}\{x\}
\end{align*}
\]

Now $\mathcal{L}\left\{\frac{dx}{dt}\right\} = sX(s) - x(0) = sX(s)$, $\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s) - 1 = sY(s) - 1$

So we have the system

\[
\begin{align*}
sX &= -X + Y \\
sY - 1 &= 2X
\end{align*}
\]

Cleaning up:

\[
\begin{align*}
(s + 1)X - Y &= 0 \\
-2X + sY &= 1
\end{align*}
\]

From the first equation we get

\[
Y = (s + 1)X
\]

Substituting this into the second equation:

\[
-2X + s(s + 1)X = 1
\]

Factoring out $X$:

\[
(-2 + s(s + 1))X = 1
\]
Solving for $X$, we get

$$X = \frac{1}{s^2 + s - 2} = \frac{1}{(s + 2)(s - 1)}.$$  

Since $Y = (s + 1)X$, we have

$$Y = \frac{s + 1}{(s + 2)(s - 1)}.$$  

So we found

$$X(s) = \frac{1}{(s + 2)(s - 1)} , \quad Y(s) = \frac{s + 1}{(s + 2)(s - 1)}.$$  

Finally, we need to find $x(t)$ and $y(t)$.

$$x(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + s - 2} \right\}, \quad y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s + 1} \right\}.$$  

By partial fractions:

$$\frac{1}{(s + 2)(s - 1)} = \frac{A}{s + 2} + \frac{B}{s - 1}, \quad \frac{1}{(s + 2)(s - 1)} = \frac{A(s - 1) + B(s + 2)}{(s + 2)(s - 1)}.$$  

So $1 = A(s - 1) + B(s + 2)$. When $s = 1, 1 = B \cdot (1 + 2)$ so $B = \frac{1}{3}$. When $s = -2, 1 = A(-2 - 1)$ so $A = -\frac{1}{3}$. So

$$X = -\frac{1}{3} \cdot \frac{1}{s + 2} + \frac{1}{3} \cdot \frac{1}{s - 1}.$$  

So

$$x(t) = \mathcal{L}^{-1}\left\{ -\frac{1}{3} \cdot \frac{1}{s + 2} + \frac{1}{3} \cdot \frac{1}{s - 1} \right\} = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t.$$  

As for $Y(s)$:

$$\frac{s + 1}{(s + 2)(s - 1)} = \frac{A}{s + 2} + \frac{B}{s - 1}, \quad \frac{s + 1}{(s - 2)(s + 1)} = \frac{A(s - 1) + B(s + 2)}{(s + 2)(s - 1)}.$$  

So $s + 1 = A(s - 1) + B(s + 2)$. When $s = 1, 1 + 1 = B \cdot (1 + 2)$ so $B = \frac{2}{3}$. When $s = -2, -2 + 1 = A(-2 - 1)$ so $A = \frac{1}{3}$. So

$$Y = \frac{1}{3} \cdot \frac{1}{s + 2} + \frac{2}{3} \cdot \frac{1}{s - 1}.$$  

So

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{3} \cdot \frac{1}{s + 2} + \frac{2}{3} \cdot \frac{1}{s - 1} \right\} = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t.$$  

Ans. to 6.1: $x(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t, \quad y(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t.$

Problem 6.2: Solve the system

$$\frac{d^2x}{dt^2} = \frac{1}{2}(-x + y) ,$$  

$$\frac{d^2y}{dt^2} = \frac{1}{2}(x - y) ,$$  

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\[ x(0) = 0 \quad , \quad x'(0) = 2 \quad ; \]
\[ y(0) = 0 \quad , \quad y'(0) = 0 \quad . \]

**Solution:** Let \( L\{x\} = X(s) \), \( L\{y\} = Y(s) \). Applying \( L \) to both equations we get

\[
L\left(\frac{d^2 x}{dt^2}\right) = \frac{1}{2} L\{x\} + \frac{1}{2} L\{y\},
\]
\[
L\left(\frac{d^2 y}{dt^2}\right) = \frac{1}{2} L\{x\} - \frac{1}{2} L\{y\},
\]

Now \( L\left(\frac{d^2 x}{dt^2}\right) = s^2 X(s) - x(0)s - x'(0) = s^2 X(s) - 2 \), \( L\left(\frac{d^2 y}{dt^2}\right) = s^2 Y(s) - y(0)s - y'(0) = s^2 Y(s) \)

So we have the system

\[
s^2 X - 2 = \frac{1}{2} X + \frac{1}{2} Y
\]
\[
s^2 Y = \frac{1}{2} X - \frac{1}{2} Y
\]

Cleaning up:

\[
(s^2 + \frac{1}{2}) X - \frac{1}{2} Y = 2 ,
\]
\[
(s^2 + \frac{1}{2}) Y = \frac{1}{2} X .
\]

From the second equation we get

\[
X = (2s^2 + 1) Y .
\]

Substituting this into the first equation:

\[
(s^2 + \frac{1}{2})(2s^2 + 1) Y - \frac{1}{2} Y = 2 ,
\]

Factor out \( Y \):

\[
(2s^4 + 2s^2 + \frac{1}{2}) Y = 2 ,
\]
\[
(2s^4 + 2s^2) Y = 2 ,
\]
\[
2s^2(s^2 + 1) Y = 2 ,
\]
So
\[
Y = \frac{1}{s^4(s^2 + 1)}
\]

Going back to \( X \):

\[
X = \frac{2s^2 + 1}{s^4(s^2 + 1)}
\]

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Finally, we need to find \( x(t) \) and \( y(t) \).

\[
x(t) = \mathcal{L}^{-1}\{ \frac{2s^2 + 1}{s^2(s^2 + 1)} \}, \quad y(t) = \mathcal{L}^{-1}\{ \frac{1}{s^2(s^2 + 1)} \}
\]

By partial fractions:

\[
X = \frac{2s^2 + 1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2 + 1}
\]

\[
\frac{2s^2 + 1}{s^2(s^2 + 1)} = \frac{(As + B)(s^2 + 1) + Cs^2}{s^2(s^2 + 1)}, \quad \text{so}
\]

\[
2s^2 + 1 = (As + B)(s^2 + 1) + Cs^2 = As^3 + Bs^2 + As + B + Cs^2 = As^3 + (B + C)s^2 + As + B
\]

Comparing coefficients, we get:

\[
A = 0, \quad B + C = 2, \quad A = 0, \quad B = 1
\]

so \( A = 0, B = 1, C = 1 \) and we have

\[
X = \frac{1}{s^2} + \frac{1}{s^2 + 1}
\]

So

\[
x(t) = \mathcal{L}^{-1}\{ \frac{1}{s^2} + \frac{1}{s^2 + 1} \} = t + \sin t
\]

As for \( Y(s) \):

\[
Y = \frac{1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2 + 1}
\]

\[
\frac{1}{s^2(s^2 + 1)} = \frac{(As + B)(s^2 + 1) + Cs^2}{s^2(s^2 + 1)}, \quad \text{so}
\]

\[
1 = (As + B)(s^2 + 1) + Cs^2 = As^3 + Bs^2 + As + B + Cs^2 = As^3 + (B + C)s^2 + As + B
\]

Comparing coefficients, we get:

\[
A = 0, \quad B + C = 0, \quad A = 0, \quad B = 1
\]

so \( A = 0, B = 1, C = -1 \) and we have

\[
Y = \frac{1}{s^2} - \frac{1}{s^2 + 1}
\]

So

\[
y(t) = \mathcal{L}^{-1}\{ \frac{1}{s^2} - \frac{1}{s^2 + 1} \} = t - \sin t
\]

**Ans. to 6.2:** \( x(t) = t + \sin t, \quad y(t) = t - \sin t. \)