1. (a) Find the complex Fourier series of $f(x) = 4x$ on the interval $-2\pi < x < 2\pi$.

**Sol. of a:** Here the interval is not $(-\pi, \pi)$ but $(-2\pi, 2\pi)$ so $p = 2\pi$, and we must transform it using $g(x) = f(xp/\pi) = f(x(2\pi)/\pi) = f(2x)$.

(For future reference, $f(x) = g(x/2)$).

So $g(x) = f(2x) = 4(2x) = 8x$, defined on the fundamental interval $(-\pi, \pi)$. So

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)e^{-inx} \, dx, \quad n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots.$$ 

In this problem $g(x) = 8x$, so

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} 8xe^{-inx} \, dx = \frac{4}{\pi} \int_{-\pi}^{\pi} xe^{-inx} \, dx.$$ 

From the formula sheet: (if $c \neq 0$, or else you get nonsense).

$$\int xe^{cx} \, dx = \frac{-1 + cx}{c^2} e^{cx} + C.$$ 

So

$$\int_{-\pi}^{\pi} xe^{-inx} \, dx = \frac{-1 + (-in)x}{(-in)^2} e^{-inx} \bigg|_{-\pi}^{\pi}.$$ 

Simplifying, using that $i^2 = -1$, we get that this equals:

$$\frac{-1 - inx}{-n^2} e^{-in\pi} \bigg|_{-\pi}^{\pi} = \frac{1 + inx}{n^2} e^{-in\pi} - \frac{1}{n^2} e^{in\pi}.$$ 

Using the famous identities $e^{i\pi} = -1$ and $e^{-i\pi} = -1$, both $e^{in\pi}$ and $e^{-in\pi}$ equal $(-1)^n$ so the above equals:

$$\left(\frac{1 + in\pi}{n^2} - \frac{1 - in\pi}{n^2}\right)(-1)^n.$$ 

Going back to $c_n$:

$$c_n = \frac{4}{\pi} \int_{-\pi}^{\pi} xe^{-inx} \, dx = \frac{8i(-1)^n}{n}.$$
But this is only correct when $n \neq 0$. We need to treat $c_0$ separately:

$$c_0 = \frac{4}{\pi} \int_{-\pi}^{\pi} x \, dx = \frac{4}{\pi} \frac{x^2}{2} \bigg|_{-\pi}^{\pi} = \frac{4}{\pi} \frac{(-\pi)^2 - \pi^2}{2} = 0 \ .$$

Going back to $g(x)$, we have

$$g(x) = \sum_{n=-\infty, n \neq 0}^{\infty} \frac{8i(-1)^n}{n} e^{inx} \ .$$

**Finally**, we must go back to $f(x)$. $f(x) = g(x/2)$. Replacing $x$ by $x/2$ right above gives

$$f(x) = \sum_{n=-\infty, n \neq 0}^{\infty} \frac{8i(-1)^n}{n} e^{inx/2} \ .$$

**Ans. to 1(a):** The complex Fourier series of $f(x) = 4x$ on the interval $-2\pi < x < 2\pi$ is

$$\sum_{n=-\infty, n \neq 0}^{\infty} \frac{8i(-1)^n}{n} e^{inx/2} \ .$$

**(b) Find the Frequency Spectrum.**

**Sol.:** Here $\omega = \pi/p = \frac{1}{2}$. Also

$$|c_n| = \left| \frac{8i(-1)^n}{n} \right| = \frac{8}{|n|} \quad (n \neq 0) \ .$$

(Since $|i| = 1$ and $|(-1)^n| = 1$). Since the frequency spectrum is $(n \omega, |c_n|), n = 0, \pm 1, \pm 2, \ldots$, we have

$$\{(\frac{n}{2}, \frac{8}{|n|})\} \quad (n \neq 0) \ , (0,0) \ .$$

**Ans. to 1(b):** The frequency spectrum of $f(x) = 4x$ in $(-2\pi, 2\pi)$ is $\{(\frac{n}{2}, \frac{8}{|n|})\} \quad (n \neq 0)$ together with $(0,0)$.

**Comments:**

1. Only about $20\%$ got both completely, but many came close, and would have done it if they had more time.

2. Another way is to use the formula for the general interval $(-p,p)$. 