Solutions to the Attendance Quiz for Lecture 13

1. Find product solutions, if possible, to the partial differential equation

\[ 2 \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial y} = 0. \]

**Sol.:** We write

\[ u(x, y) = X(x)Y(y), \]

where \( X(x) \) is a function of only \( x \) and \( Y(y) \) is a function of only \( y \). Entering it in the pde:

\[ 0 = 2 \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial y} = 2 \frac{\partial (XY)}{\partial x} + 3 \frac{\partial (XY)}{\partial y} = 2X_xY + 3XY_y. \]

So

\[ 2X'(x)Y(y) + 3X(x)Y'(y) = 0. \]

Dividing by \( XY \):

\[ \frac{2X'(x)Y(y) + 3X(x)Y'(y)}{XY} = 0. \]

Algebra:

\[ \frac{2X'}{X} + \frac{3Y'}{Y} = 0. \]

So:

\[ \frac{2X'}{X} = -\frac{3Y'}{Y}. \]

The left side **does not** depend on \( y \) and the right side **does not** depend on \( x \). But they are **equal**!

So **neither side** depends on \( x \) and \( y \). So the are **both** equal to the **same constant**, let’s call it \( k \).

We now have two odes:

\[ \frac{2X'}{X} = k. \]
\[ -\frac{3Y'}{Y} = k. \]

So

\[ \frac{X'}{X} = k/2. \]
\[ \frac{Y'}{Y} = -k/3. \]

These are the same as

\[ X''(x) - \frac{k}{2}X(x) = 0. \]
\[ Y''(y) + \frac{k}{3}Y(y) = 0. \]

The **general solutions** are

\[ X(x) = c_1e^{(k/2)x}, \]
\[
Y(y) = c_2 e^{-(k/3)y}.
\]

Going back to \( u(x, y) = X(x)Y(y) \), we (almost) finally get
\[
u(x, y) = c_1 c_2 e^{(k/2)x} e^{-(k/3)y} = C e^{k(x/2 - y/3)},
\]
where we write \( C = c_1 c_2 \) (\( c_1, c_2 \) are arbitrary constants, so \( C \), their product is yet another arbitrary constant).

**Ans. to 1:** A product solution of the pde is \( u(x, y) = C e^{k(x/2 - y/3)} \), where \( C \) and \( k \) are any constants.

**Comment:** About 30% of the people got it completely right. A common error was \( e^{k(x/2 + y/3)} \). Watch out for the sign, and review how to solve simple odes like these ones.

2. Check that \( u_1(x, y) = 2 \sin(x + y) + 3e^{x+y} \) is a solution of
\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.
\]

**Sol.** Since the pde is homogeneous (the right side is 0), by the superposition principle we can check each piece separately.

**First piece:** \( 2 \sin(x + y) \)
\[
\frac{\partial^2 (2 \sin(x + y))}{\partial x^2} - \frac{\partial^2 (2 \sin(x + y))}{\partial y^2} = (2 \sin(x + y))_{xx} - (2 \sin(x + y))_{yy}.
\]

By the chain rule \( (2 \sin(x + y))_{x} = 2 \cos(x + y) \), so \( (2 \sin(x + y))_{xx} = (2 \cos(x + y))_{x} = -2 \sin(x + y) \). Similarly: \( (2 \sin(x + y))_y = 2 \cos(x + y) \), so \( (2 \sin(x + y))_{yy} = (2 \cos(x + y))_y = -2 \sin(x + y) \). So:
\[
\frac{\partial^2 (2 \sin(x + y))}{\partial x^2} - \frac{\partial^2 (2 \sin(x + y))}{\partial y^2} = -2 \sin(x + y) - (-2 \sin(x + y)) = 0.
\]

**Second piece:**
\[
\frac{\partial^2 (3e^{x+y})}{\partial x^2} - \frac{\partial^2 (3e^{x+y})}{\partial y^2} = (3e^{x+y})_{xx} - (3e^{x+y})_{yy}.
\]

By the chain rule \( (3e^{x+y})_x = 3e^{x+y} \), so \( (3e^{x+y})_{xx} = (3e^{x+y})_{x} = 3e^{x+y} \). Similarly: \( (3e^{x+y})_y = 3e^{x+y} \), so \( (3e^{x+y})_{yy} = (3e^{x+y})_{y} = 3e^{x+y} \). So:
\[
\frac{\partial^2 (3e^{x+y})}{\partial x^2} - \frac{\partial^2 (3e^{x+y})}{\partial y^2} = 3e^{x+y} - 3e^{x+y} = 0.
\]

It follows that \( u_1(x, y) = 2 \sin(x + y) + 3e^{x+y} \), the sum of the two pieces is also a solution of the given pde.

**Comment:** Most people (who had time) got it right.