Solutions to the Attendance Quiz for Lecture 19

1.: Find the Laplace Transform of the pde $u_{xx} = 4u_{tt}$, $t > 0$.

**Sol. of 1:** Let $U(x, s) = \mathcal{L}\{u(x, t)\}$, and apply $\mathcal{L}$ to the pde getting

$$\mathcal{L}\{u_{xx}\} = 4\mathcal{L}\{u_{tt}\}$$

So

$$U''(x, s) = 4(s^2U(x, s) - su(x, 0) - u_t(x, 0)) = 4s^2U(x, s) - 4su(x, 0) - 4u_t(x, 0))$$

Cleaning up, we get the ode

$$U''(x, s) - 4s^2U(x, s) = -4su(x, 0) - 4u_t(x, 0).$$

2.: Solve the pde

$$u_{xx} = u_{tt}, \quad 0 < x < 1, \quad t > 0,$$

subject to the **boundary-conditions**

$$u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t > 0,$$

and the **initial conditions**

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin(\pi x/2), \quad 0 < x < 1.$$

**Sol. to 2:** Let $U(x, s) = \mathcal{L}\{u(x, t)\}$. Apply $\mathcal{L}$ to the pde

$$\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}.$$

Using the ‘dictionary’:

$$U''(x, s) = s^2U(x, s) - su(x, 0) - u_t(x, 0).$$

Using the initial conditions, we get the inhomogeneous ode (for the sake of brevity we rewrite $U(x, s)$ as $U(x)$)

$$U''(x) - s^2U(x) = -\sin(\pi x/2).$$

Translating the boundary conditions gives

$$U'(0) = 0, \quad U'(1) = 0.$$

The general solution of the inhomogeneous version is:

$$U''(x) - s^2U(x) = 0.$$
is
\[ U(x) = c_1 e^{sx} + c_2 e^{-sx} \]
(Since the auxiliary equation is \( r^2 - s^2 = 0 \) whose roots are \( r = -s \) and \( r = s \)).

A template for a particular solution, in general, would be
\[ U(x) = A \sin(\pi x/2) + B \cos(\pi x/2) \]
but since \( U'(x) \) does not show up, it is safe to take the simpler template
\[ U(x) = A \sin(\pi x/2) \]
So \( U''(x) = -A(\pi/2)^2 \sin(\pi x/2) \). Plugging into the ode, we get
\[-A(\pi/2)^2 \sin(\pi x/2) - s^2 A \sin(\pi x/2) = -\sin(\pi x/2)\]
Simplifying:
\[-A \sin(\pi x/2)((\pi/2)^2 + s^2) = -\sin(\pi x/2)\]
Dividing by \(-\sin(\pi x/2)\):
\[ A((\pi/2)^2 + s^2) = 1 \]
yielding:
\[ A = \frac{1}{(\pi/2)^2 + s^2} \]
So the general solution of the ode is
\[ U(x) = c_1 e^{sx} + c_2 e^{-sx} + \frac{1}{(\pi/2)^2 + s^2} \sin(\pi x/2) \]
In order to find \( c_1 \) and \( c_2 \), we must use the boundary condition \( U'(0) = 0 \) and \( U'(1) = 0 \). But first we need \( U'(x) \):
\[ U'(x) = c_1 se^{sx} - c_2 se^{-sx} + \frac{1}{(\pi/2)^2 + s^2} \frac{\pi}{2} \sin(\pi x/2) \]
We get
\[ U'(0) = c_1 s - c_2 s + 0 \]
\[ U'(1) = c_1 se^s - c_2 se^{-s} + 0 \]
Since \( U'(0) = 0 \) and \( U'(1) = 0 \) we get the system of two equations with two unknowns:
\[ c_1 - c_2 = 0 \quad , \quad c_1 e^s - c_2 e^{-s} = 0 \]
whose solution is \( c_1 = 0 \) and \( c_2 = 0 \). Going back to the general solution, we found a formula for \( U(x, s) \):
\[ U(x, s) = \frac{1}{(\pi/2)^2 + s^2} \sin(\pi x/2) \]
Finally, we apply \( L^{-1} \) getting (using the fact from the table that \( L^{-1}\frac{1}{(\pi/2)^2 + s^2} = \frac{1}{\pi} \sin(\pi t) \)):
\[ u(x, t) = L^{-1}\left(\frac{1}{(\pi/2)^2 + s^2}\right) = \sin(\pi x/2) L^{-1}\left(\frac{1}{(\pi/2)^2 + s^2}\right) = \sin(\pi x/2) \sin(\pi t) \]
\[ = \frac{2}{\pi} \sin(\pi x/2) \sin(\pi t) \]
Ans. to 2: \( u(x, t) = \frac{2}{\pi} \sin(\pi x/2) \sin(\pi t) \).