Solutions to the Attendance Quiz 1

1. Using the definition find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of

$$f(t) = \begin{cases} 3, & \text{if } 0 \leq t \leq 1; \\ e^t, & \text{if } t \geq 1. \end{cases}$$

Sol.: By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt .$$

Since this is a discontinuous function defined differently in different intervals, we have to break-up the integration from 0 to $\infty$ to the two parts: from 0 to 1, and from 1 to $\infty$:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt = \int_0^1 e^{-st} f(t) \, dt + \int_1^\infty e^{-st} f(t) \, dt$$

Now, at each of the two integrals, we replace $f(t)$ by the appropriate expression:

$$= \int_0^1 e^{-st} \cdot 3 \, dt + \int_1^\infty e^{-st} \cdot e^t .$$

The first integral is a proper integral, since the limits of integration are finite. Using the famous formula

$$\int e^{ct} \, dt = \frac{e^{ct}}{c} + C ,$$

we have

$$\int_0^1 e^{-st} \cdot 3 \, dt = 3 \int_0^1 e^{-st} \, dt = \frac{3e^{-st}}{-s} \bigg|_0^1 = \frac{-3}{s}(e^{-s} - e^{-0}) = \frac{-3}{s}(e^{-s} - 1) = \frac{3(1 - e^{-s})}{s} .$$

Now we go to the second, improper, integral: (assume that $s > 1$, as we may)

$$\int_1^\infty e^{-st} \cdot e^t = \int_1^\infty e^{(1-s)t} = \int_1^\infty \frac{e^{(1-s)t}}{1-s} \, dt$$

$$= \frac{e^{(1-s)\cdot \infty}}{1-s} \bigg|_1^{\infty} = \frac{e^{(1-s)\cdot \infty}}{1-s} - \frac{e^{(1-s)\cdot 1}}{1-s} = e^{-\infty} - e^{(1-s)} = 0 - e^{(1-s)} = \frac{e^{(1-s)}}{s-1} .$$

Adding the two pieces up, we get

Ans. to 1: $\mathcal{L}\{f(t)\} = \frac{3(1 - e^{-s})}{s} + \frac{e^{(1-s)}}{s-1} .$

Comments: Only about %30 of the students finished it completely correctly, but about %80 were on the right track. Many people had trouble with the improper integral.

2. Using Tables, find $\mathcal{L}\{f(t)\}$, if $f(t) = (t + 1)(t + 2) + e^t + \sin t$.

Sol.: First we expand:

$$f(t) = t^2 + 3t + 2 + e^t + \sin t .$$
Next we apply $\mathcal{L}$:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 3t + 2 + e^t + \sin t\} \ .$$

Using the linearity property of $\mathcal{L}$, this becomes

$$\mathcal{L}\{t^2\} + 3\mathcal{L}\{t\} + 2\mathcal{L}\{1\} + \mathcal{L}\{e^t\} + \mathcal{L}\{\sin t\} \ .$$

Now (and only now!) we use the table:

$$\mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}} \ ,$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \ ,$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2} \ .$$

So

$$\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3} \ ,$$

$$\mathcal{L}\{t\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2} \ ,$$

$$\mathcal{L}\{1\} = \frac{0!}{s^{0+1}} = \frac{1}{s} \ ,$$

$$\mathcal{L}\{e^t\} = \frac{1}{s-1} \ ,$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1^2} = \frac{1}{s^2 + 1} \ .$$

Putting these together, we get: **Ans. to 2:**

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s} + \frac{1}{s-1} + \frac{1}{s^2 + 1} \ .$$

**Comment:** About 80% of the students got it completely right. Some people forgot what $k!$ means! $k!$ means $1 \cdot 2 \cdots k$, so $1!$ is 1, $2!$ is 2, $3!$ is 6 etc. A couple of people had $k$ in the answer. This is wrong! $k$ only features in the table, and one has to find the applicable $k$.