Solutions to the Attendance Quiz for Lecture 6

1. \[
\frac{dx}{dt} = -x + y , \\
\frac{dy}{dt} = 2x , \\
x(0) = 0 , \quad y(0) = 3 .
\]

**Sol.** We first apply the Laplace transform, and a always, write \( X = \mathcal{L}\{x\}, Y = \mathcal{L}\{y\} \).

\[
\mathcal{L}\{x'(t)\} = \mathcal{L}\{-x + y\} , \\
\mathcal{L}\{y'(t)\} = \mathcal{L}\{2x\} .
\]

Since \( \mathcal{L}\{x'\} = sX - x(0) = sX \) and \( \mathcal{L}\{y'\} = sY - y(0) = sY - 3 \), we have:

\[
sX = -X + Y , \\
sY - 3 = 2X .
\]

Rearranging in standard form:

\[
(s + 1)X - Y = 0 , \\
-2X + sY = 3 .
\]

From the first equation, we have \( Y = (s + 1)X \). Plugging this into the second equation we get

\[
-2X + s(s + 1)X = 3 .
\]

Factoring out \( X \):

\[
(s(s + 1) - 2)X = 3 ,
\]

Expanding:

\[
(s^2 + s - 2)X = 3 ,
\]

Factorizing:

\[
(s - 1)(s + 2)X = 3 ,
\]

Solving for \( X \):

\[
X = \frac{3}{(s - 1)(s + 2)} .
\]

Back-substitution:

\[
Y = (s + 1)X = \frac{3(s + 1)}{(s - 1)(s + 2)} .
\]

Now it is time to apply \( \mathcal{L}^{-1} \).

\[
x(t) = \mathcal{L}^{-1}\left\{\frac{3}{(s - 1)(s + 2)}\right\} , \quad y(t) = \mathcal{L}^{-1}\left\{\frac{3s + 3}{(s - 1)(s + 2)}\right\} .
\]
Partial fraction for $X$, we use the following template

$$\frac{3}{(s-1)(s+2)} = A\frac{1}{s-1} + B\frac{1}{s+2}.$$ 

Common denominator:

$$\frac{3}{(s-1)(s+2)} = \frac{A(s+2) + B(s-1)}{(s-1)(s+2)}.$$ 

Equating the tops: $3 = A(s+2) + B(s-1)$. Convenient values: $s = -2$ gives $3 = A(-2 + 2) + B(-2 - 1)$ so $B = -1$; $s = 1$ gives $3 = A(1 + 2) + B(1 - 1)$, so $3 = 3A$, so $A = 1$, going back to the template:

$$\frac{3}{(s-1)(s+2)} = \frac{1}{s-1} - \frac{1}{s+2}.$$ 

And, so

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+2}\right\} = e^t - e^{-2t}.$$ 

Now we do the same thing for $Y = \frac{3(s+1)}{(s-1)(s+2)}$. We must use the following template

$$\frac{3s + 3}{(s-1)(s+2)} = A\frac{1}{s-1} + B\frac{1}{s+2}.$$ 

Common denominator:

$$\frac{3s + 3}{(s-1)(s+2)} = \frac{A(s+2) + B(s-1)}{(s-1)(s+2)}.$$ 

Equating the tops: $3s + 3 = A(s+2) + B(s-1)$. Convenient values: $s = -2$ gives $3(-2) + 3 = A(-2 + 2) + B(-2 - 1)$ so $B = 1$; $s = 1$ gives $3(1) + 3 = A(1 + 2) + B(1 - 1)$, so $6 = 3A$, so $A = 2$, going back to the template:

$$\frac{3}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{1}{s+2}.$$ 

And, so

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{1}{s+2}\right\} = 2e^t + e^{-2t}.$$ 

Ans. to 1: $x(t) = e^t - e^{-2t}$, $y(t) = 2e^t + e^{-2t}$.

Comment: About %40 of the students did it perfectly. Another %40 know how to do it, but messed up sooner or later, or ran out of time.