Solutions to the Attendance Quiz for Lecture 8

1 Find the Fourier series of \( f(x) = 2x \) on the interval \((-2\pi, 2\pi)\).

First Sol. (By transforming to the interval \((-\pi, \pi)\) that has nicer-looking formulas)

We transform the problem to the standard interval, \((-\pi, \pi)\), by considering
\[
g(x) = f(2x) = 2(2x) = 4x
\]
that is defined on \((-\pi, \pi)\). At the end, once we get the answer for \(g(x)\), we go back to \(f(x)\) with the reverse relation
\[
f(x) = g(x/2)
\]
In this problem
\[
g(x) = f(2x) = 2(2x) = 4x
\]
We use the formulas
\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx,
\]
\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,
\]
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]
Since \(f(x)\) is an odd function, \(a_n\) is automatically 0, so we shouldn’t bother with it. Also \(a_0\) is 0. We only have to worry about \(b_n\).

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} 4x \sin nx \, dx.
\]
Now we use the formula from the formula sheet
\[
\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2},
\]
So
\[
b_n = \frac{4}{\pi} \cdot \frac{\pi}{n^2} - \frac{4(-1)^n}{n^2} = -\frac{4(-1)^n}{n^2}.
\]
So the Fourier Series of \(g(x)\) is
\[
g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = 0 + 0 + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n} \sin nx = -8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.
\]
Finally: going back to $f(x)$, using $f(x) = g(x/2)$, we get that the Fourier Series of $f(x) = 2x$ is:

$$-8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{nx}{2}.$$

Ans. to 1.: The Fourier Series of $f(x) = 2x$ over the interval $(-2\pi, 2\pi)$ is $-8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{nx}{2}$.

Second Sol. (By using the more complicated formulas for a general interval $(-L, L)$).

We use the formulas

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx.$$

Since $f(x)$ is an odd function, $a_n$ is automatically 0, so we shouldn’t bother with it. Also (once again because $f(x)$ is odd) $a_0$ is 0. So we only have to worry about $b_n$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} 2x \sin \left( \frac{n\pi x}{2\pi} \right) \, dx = \frac{1}{\pi} \int_{-2\pi}^{2\pi} x \sin \left( \frac{n}{2} x \right) \, dx$$

Now we use the formula from the formula sheet

$$\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}.$$

So

$$b_n = \frac{1}{\pi} \left( \frac{\sin((n/2)x) - (n/2)x \cos((n/2)x)}{(n/2)^2} \right) \bigg|_{-2\pi}^{2\pi}$$

$$= \frac{1}{\pi} \left( \frac{\sin((n/2)(2\pi)) - (n/2)(2\pi) \cos((n/2)(2\pi))}{(n/2)^2} \right) - \frac{1}{\pi} \left( \frac{\sin(n/2)(-2\pi) - (n/2)(-2\pi) \cos((n/2)(-2\pi))}{(n/2)^2} \right)$$

$$= \frac{1}{\pi} \left( \frac{\sin(n\pi) - (n/2)(2\pi) \cos((n\pi))}{(n/2)^2} \right) - \frac{1}{\pi} \left( \frac{\sin(-n\pi) - (n/2)(2\pi) \cos((-n\pi))}{(n/2)^2} \right)$$

$$= \frac{1}{\pi} \left( \frac{0 - n\pi(-1)^n}{(n/2)^2} \right) - \frac{1}{\pi} \left( \frac{0 - n\pi(-1)^n}{(n/2)^2} \right)$$

$$= \frac{4}{\pi n^2} (n\pi)(-1)^n - \frac{4}{\pi n^2} (n\pi)(-1)^n = \frac{-8}{n} \cdot (-1)^n$$

So, once again

Ans. to 1.: The Fourier Series of $f(x) = 2x$ over the interval $(-2\pi, 2\pi)$ is

$$-8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi}{2}.$$

Comment: Few people got it completely, but many were on the right track. This is an important type of problem. Please study it carefully.