1. Find the Fourier series of \( f(x) = 2x^2 + 1 \) on the interval \((−\pi, \pi)\).

**Note:** You may use the ready-made indefinite integrals:

\[
\int x^2 \sin nx \, dx = \frac{(2 - n^2 x^2) \cos(nx) + 2nx \sin(nx)}{n^3}
\]

\[
\int x^2 \cos nx \, dx = \frac{(n^2 x^2 - 2) \sin(nx) + 2xn \cos(x)}{n^3}
\]

**Sol.** The Fourier series of 1 is 1 and the Fourier series of \( 2x^2 \) is twice that of \( x^2 \), so let’s first find the Fourier series of \( x^2 \).

The Fourier Series, in general, is:

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \]

where

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx , \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx , \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx .
\]

Since \( x^2 \) is an **even** function, we can ignore \( b_n \) (they are automatically zero).

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{x^3}{3\pi} \bigg|_{-\pi}^{\pi} = \frac{2\pi^2}{3}.
\]

Now, by the formula

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{\frac{(n^2 x^2 - 2) \sin(nx) + 2xn \cos(nx)}{n^3}}{\pi n^3} \bigg|_{-\pi}^{\pi} = \frac{0 + 2\pi n (-1)^n}{\pi n^3} - \frac{0 + 2(-\pi) n (-1)^n}{\pi n^3} = \frac{4\pi n (-1)^n}{\pi n^3} = \frac{4(-1)^n}{n^2}.
\]

So the Fourier series of \( x^2 \) is

\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx .
\]

**Finally** to get the Fourier series of \( 2x^2 + 1 \) we multiply by 2 and add 1:

\[
2x^2 + 1 = 1 + \frac{2\pi^3}{3} + 8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx .
\]
This is the ans.

Comments

1. An even more efficient way, since \( x^2 \) (and also \( 1 + 2x^2 \)) is an even function (note the pun), is to use

\[
a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx.
\]

2. Many people gave the answer

\[
\frac{2\pi^3}{3} + 8 \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \cos nx + 1.
\]

They got full credit, but strictly speaking, the +1 should be together with the other number \( \frac{2\pi^3}{3} \). The pure number should stand in front.

3. Some people got the correct answer and then “improved” it by writing

\[
1 + \frac{2\pi^3}{3} + 8 \sum_{k=0}^\infty \frac{(-1)^n}{(2k+1)^2} \cos(2k+1)x.
\]

This is wrong! (and they lost points for doing such nonsense. They blindly aped the simplification in one of the homework problem where there was \( 1 - (-1)^n \) at the top, but \( (-1)^n \) is never zero, so it is completely a different situation!

If in doubt, do not “simplify” (and take a chance of producing nonsense that will cost you important points (or your job, in real life).