1. (a) (8 points) Find the complex Fourier series \( f(x) = x \) on the interval \(-\pi < x < \pi\). (b) (2 points) Find its frequency spectrum.

Sol. to a):

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{-inx} \, dx.
\]

When \( n = 0 \), this is easy:

\[
c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = \frac{1}{2\pi} \left( \frac{x^2}{2} \bigg|_{-\pi}^{\pi} \right) = \frac{1}{2\pi} \left( \frac{\pi^2 - (-\pi)^2}{2} \right) = 0.
\]

Remember (integration by parts, or use the cheatsheet)

\[
\int x e^{cx} \, dx = \left( -\frac{1}{c^2} + \frac{x}{c} \right) e^{cx}
\]

Here \( c = -in \) \((n \neq 0)\) so:

\[
\int_{-\pi}^{\pi} xe^{-inx} \, dx = \left( \frac{1}{(-in)^2} + \frac{x}{-in} \right) e^{-inx} \bigg|_{-\pi}^{\pi} = \left( \frac{-1}{(-in)^2} + \frac{ix}{n} \right) e^{-inx} \bigg|_{-\pi}^{\pi}
\]

\[
= \left( \frac{1}{n^2} + \frac{i\pi}{n} \right) e^{-in(\pi)} - \left( \frac{1}{n^2} + \frac{i(-\pi)}{n} \right) e^{i\pi(n)}
\]

Now \( e^{i\pi} = -1 \), so \( e^{in\pi} = (-1)^n \). and also \( e^{-inx} = (-1)^n \). So this equals

\[
\left( \frac{1}{n^2} + \frac{i\pi}{n} \right) (-1)^n - \left( \frac{1}{n^2} + \frac{i(-\pi)}{n} \right) (-1)^n = \frac{2i\pi(-1)^n}{n}.
\]

Multiplying by \( \frac{1}{2\pi} \), we get:

\[
c_n = \frac{1}{2\pi} \frac{2i\pi(-1)^n}{n} = \frac{i(-1)^n}{n}
\]

Ans. to 1a: The Fourier Series of \( f(x) = x \) on the interval \((-\pi, \pi)\) is:

\[
\sum_{n=-\infty, n\neq 0}^{\infty} \frac{i(-1)^n}{n} e^{inx}.
\]

Comment: Most people stated it right, but only about \(\%40\) got it fully correct. Some people need to review their algebra, and how to handle complex numbers. Remember: \(i^2 = -1, i^3 = -i, i^4 = 1, 1/i = -i\).

Sol. of 1b: Recall that the frequency spectrum of a function \( f(x) \) on the interval \((-p, p)\) is the infinite (double) sequence of points:

\[
(n\omega, |c_n|), \quad n = 0, \pm 1, \pm 2, \ldots
\]
Where $\omega = \pi/p$.

Here $p = \pi$ so $\omega = \pi/p = \pi/\pi = 1$, so the **frequency spectrum** is

$$(n, |c_n|), n = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

When $n \neq 0$, $c_n = \frac{i(-1)^n}{n}$, so $|c_n| = \frac{1}{|n|}$, (since $|i| = 1$ and $|(-1)^n| = 1$). When $n = 0$, $c_0 = 0$, so we get $(0,0)$.

So **Ans. to 1b**: The frequency spectrum of $f(x) = x$ on the interval $(-\pi, \pi)$ is the set

$$\{(0,0)\} \cup \{(n, \frac{1}{|n|}); n = \pm 1, \pm 2, \pm 3, \ldots\}.$$ 

**Comment**: A common error was to get $\omega$ wrong. Some people got $\omega = 2$. Watch out! Another common error was to write $(n, 1/n)$. This is wrong for two reasons. First the answer is $(n\omega, |c_n|)$. Don’t forget the absolute value. Since $n$ can be positive or negative and the second component of the energy spectrum is never negative it should be $(n, |1/n|)$. But even this is nonsense when $n = 0$, that has to be done separately. Since $c_n = 0$, one has to add the point $(0,0)$. 