1. (5 points) The Zoe polynomials $Z_n(x)$ are defined by

$$Z_n(x) = Z_{n-1}(x) + Z_{n-2}(x) + xZ_{n-3}(x)$$

with initial conditions $Z_0(x) = 1$, $Z_1(x) = x$, $Z_2(x) = x^2$. Find $Z_3(x)$ and $Z_4(x)$.

**Sol.** When $n = 3$:

$$Z_3(x) = Z_2(x) + Z_1(x) + xZ_0(x) = x^2 + x + x(1) = x^2 + 2x$$

When $n = 4$:

$$Z_4(x) = Z_3(x) + Z_2(x) + xZ_1(x) = x^2 + 2x + x^2 + x(x) = x^2 + 2x + x^2 + x^2 = 3x^2 + 2x$$

**Ans. to 1.:** $Z_3(x) = x^2 + 2x$, $Z_4(x) = 3x^2 + 2x$.

2. (5 points) Find product solutions, if possible, to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

**Sol.:** We try:

$$u(x, y) = X(x)Y(y) .$$

Now:

$$\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y) ,$$

$$\frac{\partial^2 u}{\partial x \partial y} = X'(x)Y'(y) ,$$

$$\frac{\partial^2 u}{\partial y^2} = X(x)Y''(y) .$$

Plugging this in the pde we get:

$$X''Y - 2X'Y' + XY'' = 0 .$$

Dividing by $XY$, we get:

$$\frac{X''}{X} - 2\frac{X'Y'}{XY} + \frac{Y''}{Y} = 0 .$$

Now we are stuck. There is no way to separate the $X(x)$ (and $x$) -stuff from the $Y(y)$ (and $y$) stuff.

**Ans. to 2:** The pde is inseparable. It is impossible to use separation of variables.