A Maple One-Line Proof of George Andrews's Theorem that the Number of Triangles with Integer Sides Whose Perimeter is n Equals $\{\frac{n^2}{12}\} - \lfloor \frac{n}{4} \rfloor \lfloor \frac{n+2}{4} \rfloor$

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evalb(seq(coeff(taylor(q^3/(1-q^2)/(1-q^3)/(1-q^4),q=0,37),q,i),i=0..36)
=seq(round(n^2/12)-trunc(n/4)*trunc((n+2)/4),n=0..36));
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Comments by Doron Zeilberger

1. The succinct formula of the title was discovered and first proved by George Andrews[A1], in a less-than-one-page cute note, improving a four-page note by Jordan et. al. [JWW]. Andrews's note, while short and sweet, is not self-contained, and refers to results from his classic [A2].

2. Here is a justification of the Maple one-line. Arrange the lengths of the sides of a typical integer-side triangle in non-increasing order, and write them as $[a+b+c+1, b+c+1, c+1]$ for $a, b, c \geq 0$. By [E] I.20, $(b+c+1)+(c+1) > a+b+c+1$, so $c \geq a$, so $c = a+t$ for $t \geq 0$. So a generic integer-side triangle can be written as $[a+b+(a+t)+1, b+(a+t)+1, a+t+1] = [2a+b+t+1, a+b+t+1, a+t+1]$, and hence the number of triangles with integer sides whose perimeter equals n is the coefficient of q^n in

$$\sum_{a,b,t \geq 0} q^{(2a+b+t+1)+(a+b+t+1)+(a+t+1)} = \sum_{a,b,t \geq 0} q^{4a+3t+2b+3} = q^3 \left(\sum_{a \geq 0} q^{4a}\right)\left(\sum_{t \geq 0} q^{3t}\right)\left(\sum_{b \geq 0} q^{2b}\right) =$$

$$\frac{q^3}{(1-q^4)(1-q^3)(1-q^2)} \quad .$$

The coefficient of q^n is obviously a *quasi-polynomial* of degree 2 and "period" $lcm(2,3,4) = 12$, but so is $\{\frac{n^2}{12}\} - \lfloor \frac{n}{4} \rfloor \lfloor \frac{n+2}{4} \rfloor$, hence it suffices to check that the first $(2+1) \cdot 12 + 1 = 37$ values match. $\square$

3. This is yet another example, where "physical" (as opposed to "mathematical") induction, i.e. checking **finitely** (and not that many!) special cases constitutes a *perfectly rigorous proof*! No need for the "simple" argument by "mathematical" induction alluded to by Andrews in his note. In this case we were in the *quasi-polynomial* **ansatz**. See [Z1][Z2], as well as the future classic [KP].

References

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