How Borwein and Bradley's Hypergeometric Ugly Duckling Turned Into a Beautiful 'Eulerian' Swan, thanks to Almkvist and Granville's Magic

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Every Mathematician has only a few[2] *tricks*— Gian-Carlo Rota [Notices 44(1)(1/1997), 24].

Last spring, Jon Borwein and David Bradley[BB] came up with an intriguing conjecture. They stated several equivalent statements, one of them, due to Wenchang Chu, states:

$$\sum_{k=1}^{n} \frac{2n^2}{k^2} \frac{\prod_{i=1}^{n-1}(i^4+4k^4)}{\prod_{\substack{i=1\\ i\neq k}}^{n}(k^4-i^4)} = \binom{2n}{n} \quad . \qquad (BBC)$$

Superficially, this looks like grit for the EKHAD mill. A closer look, however, shows that it is not doable *as is*, because $(k^4 - i^4)$ and $(i^4 + 4k^4)$ don't factor completely over the rational numbers. Nevertheless, since I had this nifty hammer, I was sure that this identity is yet another nail, that just needs some polishing before it can be hit. I tried lots of Rube Goldberg contraptions, but to no avail, EKHAD refused to find the 'trivializing generalization'.

My mistake was in being trapped in the hypergeometric paradigm. Gert Almkvist and Andrew Granville's beautiful proof[AG] showed that the hypergeometric appearance is but a red herring. Instead it followed from :

(i) Euler's good old $\Delta^n P(x) = 0$, where $\Delta f(x) := f(x+1) - f(x)$, provided the degree of $P(x)$ is $< n$. (The proof is trivial: Acting Δ on any polynomial $Q(x)$, reduces its degree by 1.)

(ii) The indefinite sum $P(k) := \sum_{k'=1}^{k} p(k')$ of a polynomial $p(k)$ is a polynomial of degree one higher in k. (One way to prove this trivial fact is to prove it for the basis $p_i(k) := \binom{i+k}{i}$, for which the indefinite sum is $p_{i+1}(k)$, thanks to Pascal's triangle.)

(iii) The elementary symmetric functions, $\{e_r\}$, are polynomials in the power-sum functions $\{p_r\}$. This follows from (e.g. Macdonald's *Symmetric Functions*)
$\sum_{r=0}^{n} e_r t^r = \prod_{i=1}^{n}(1+x_i t) = \exp(\log(\prod_{i=1}^{n}(1+x_i t))) = \exp(\sum_{i=1}^{n}\log(1+x_i t)) = \exp(\sum_{r=1}^{\infty}(-1)^r \frac{p_r}{r} t^r)$.

(iv) The factorizations $k^4 - i^4 = (k-i)(k+i)(k^2+i^2)$ and $i^4 + 4k^4 = ((i-k)^2+k^2)((i+k)^2+k^2)$, and cancellations in products.

Using their method, Almkvist and Granville[AG] concocted many other non-shaloshable identities.

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[2] However, 'few' should be at least 2.

Another 'master identity' that one should keep in mind when trying to prove 'strange' identities, is the Lagrange[3] Interpolation Formula: if $degP(x) < n$ then,

$$P(x) = \sum_{k=1}^{n} \frac{\prod_{\substack{i=1 \\ i\neq k}}^{n} (x - x_i)}{\prod_{\substack{i=1 \\ i\neq k}}^{n} (x_k - x_i)} P(x_k) \quad .$$

Recall the trivial proof: both sides are polynomials of degree $< n$ that match at the n values $x_1, \ldots, x_n$. By taking $P(x)$ to be, for example,

$$P(x) = \prod_{j=1}^{n-1} (x + y_j),$$

or more complicated forms, and choosing x_i and y_j in countless ways, it is possible to come up with many identities, hypergeometric or otherwise, that would stump Shalosh.

REFERENCES

[AG] Gert Almkvist and Andrew Granville, *Borwein and Bradley's Apéry-like formulae for* $\zeta(4n+3)$, preprint. `http://www.math.uga.edu/~andrew/Postscript/BorBrad.ps`.

[BB] Jon Borwein and David Bradley, *Empirically determined Apéry-like formulae for* $\zeta(4n+3)$, Experimental Math, to appear. Downloadable from CECM's Home Page `http://www.cecm.sfu.ca`.

[3] Also the Lagrange Inversion Formula, used so beautifully by Ira Gessel and Dennis Stanton, but that's another story.