A Snappy Proof That 123-Avoiding Words are Equinumerous with 132-Avoiding Words

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Define a mapping F on a word w in the alphabet $\{1, 2, \ldots, n\}$ recursively as follows. If w is empty, then $F(w) := w$. Otherwise, $i := w_1$, and let W be the word obtained from w by first beheading it, and then replacing all letters larger than $i+1$ by $i+1$, and let s be the sub-sequence of w obtained by deleting the letters $\leq i$. Let $\bar{s}$ be the reverse of s. Let $V := F(W)$, and let U be the word obtained from V by replacing (in order) the letters that are $i+1$ by the members of $\bar{s}$. Finally let $F(w) := iU$.

F is an involution that sends 123-avoiding words to 132-avoiding ones, and vice versa. This follows from the fact that s above in then non-increasing and non-decreasing respectively. Hence, for any vector of non-negative integers $(a_1, \ldots, a_n)$, amongst the $(a_1 + \ldots + a_n)!/(a_1! \cdots a_n!)$ words with a_1 1's, ..., a_n n's, the number of those that avoid the pattern 123 equals the number of those that avoid 132, a result first proved by Albert, Aldred, Atkinson, Handley and Holton (Europ. J. Comb. **22** (2001), 1021-1031). For permutations (i.e. $a_1 = \ldots = a_n = 1$), F coincides with the classical bijection of Rodica Simion and Frank Schmidt (Europ. J. Comb. **6** (1985), 386-406), but my recursive formulation (that only works because we have the extra elbow-room of words!) is more transparent.

It also follows that we have a quick recurrence that enables us to compute the number of such words, let's call it $A(a_1, \ldots, a_n)$.

$$A(a_1, \ldots, a_n) = \sum_{i=1}^{n} A(a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1} + \ldots + a_n) \quad .$$

My brilliant student, Lara Pudwell, is currently exploring extensions and ramifications.

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