

## A SHORT WZ-STYLE PROOF OF ABEL'S IDENTITY

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*Dedicated to Dominique Foata on the occasion of his sixtieth birthday.*

ABSTRACT. Using a certification procedure for Abel-type sums, we present a computerized proof of Abel's identity.

Abel's identity [1],[2] can be proved in many ways, including the elegant combinatorial methods of [3]. Here we present the first computer-generated proof using the methods introduced in [4].

**Theorem**([1]; [2], p.128). *For  $n \geq 0$ :*

$$\sum_{k=0}^n \binom{n}{k} (r+k)^{k-1} (s-k)^{n-k} = \frac{(r+s)^n}{r}. \quad (1)$$

*Proof.* Let  $F_{n,k}(r,s)$  and  $a_n(r,s)$  denote, respectively, the summand and sum on the LHS of (1), and let  $G_{n,k} := (s-n) \binom{n-1}{k-1} (k+r)^{k-1} (s-k)^{n-k-1}$ . Since

$$F_{n,k}(r,s) - sF_{n-1,k}(r,s) - (n+r)F_{n-1,k}(r+1,s-1) + (n-1)(r+s)F_{n-2,k}(r+1,s-1) = G_{n,k} - G_{n,k+1},$$

(check!), we have by summing from  $k=0$  to  $k=n$ , thanks to the telescoping on the right:

$$a_n(r,s) - sa_{n-1}(r,s) - (n+r)a_{n-1}(r+1,s-1) + (n-1)(r+s)a_{n-2}(r+1,s-1) = 0.$$

Since  $(r+s)^n \cdot r^{-1}$  also satisfies this recurrence (check!) with the same initial conditions  $a_0(r,s) = r^{-1}$  and  $a_1(r,s) = (r+s) \cdot r^{-1}$ , (1) follows.  $\square$

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### REFERENCES

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