CHU'S 1303 IDENTITY IMPLIES BOMBIERI'S 1990 NORM-INEQUALITY
[Via An Identity of Beauzamy and Dégot]

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Blessed are the meek: for they shall inherit the earth (Matthew V.5)

Inequalities are deep, while *equalities* are shallow. Nevertheless, it sometimes happens that a deep inequality, **A**, follows from a mere *equality* **B**, which, in turn, follows from a more general, and *trivial*[2] identity **C**.

In this note we demonstrate this, following [3], with **A**:= Bombieri's norm inequality[2][3], **B**:= an identity of Reznick[5], and **C** := an identity of Beauzamy and Dégot[3]. This exposition differs from the original only in the punch line: I give a 1-line proof of **C**, using Chu's identity.

Let $P(x_1, \ldots, x_n)$ and $Q(x_1, \ldots, x_n)$ be two polynomials in n variables:

$$P = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \quad , \quad Q = \sum_{i_1,\ldots,i_n \geq 0} b_{i_1,\ldots,i_n} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \quad .$$

The *Bombieri inner product*[2] is defined by

$$[P, Q] := \sum_{i_1,\ldots,i_n \geq 0} (i_1! \ldots i_n!) \cdot a_{i_1,\ldots,i_n} b_{i_1,\ldots,i_n} \quad ,$$

and the *Bombieri norm*, by: $\|P\| := \sqrt{[P, P]}$.

Bombieri's Inequality A: Let P and Q be any *homogeneous* polynomials in $(x_1, \ldots, x_n)$, then

$$\|PQ\| \geq \|P\| \|Q\| \quad .$$

In order to state **B** and **C**, we need to introduce the following notation. $D_i := \frac{\partial}{\partial x_i}$, $(i = 1, \ldots, n)$, $P^{(i_1,\ldots,i_n)} := D_1^{i_1} \ldots D_n^{i_n} P$, and for any polynomial $A(x_1, \ldots, x_n)$, $A(D_1, \ldots, D_n)$ denotes the linear partial differential operator with constant coefficients obtained by replacing x_i by D_i.

A follows almost immediately from([5][3]):

Reznick's Identity B: For any polynomials P, Q in n variables:

$$\|PQ\|^2 = \sum_{i_1,\ldots,i_n \geq 0} \frac{\|P^{(i_1,\ldots,i_n)}(D_1, \ldots, D_n) Q(x_1, \ldots, x_n)\|^2}{i_1! \cdot \ldots \cdot i_n!} \quad .$$

Beauzamy and Dégot's Identity C: For any polynomials P,Q,R,S in n variables:

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[2] Trivial to verify, not to conceive!

[3] It was needed by Beauzamy and Enflo in their research on deep questions on Banach spaces. It also turned out to have far reaching applications to computer algebra![1].

$$[PQ,RS] = \sum_{i_1,\ldots,i_n \geq 0} \frac{[R^{(i_1,\ldots,i_n)}(D_1,\ldots,D_n)Q(x_1,\ldots,x_n), P^{(i_1,\ldots,i_n)}(D_1,\ldots,D_n)S(x_1,\ldots,x_n)]}{(i_1!\ldots i_n!)} \quad .$$

Proof of B $\Rightarrow$ A: Pick the terms for which $i_1+\ldots+i_n$ equals the (total) degree of P, let's call it p, and note that for those $(i_1,\ldots,i_n)$, $P^{(i_1,\ldots,i_n)}(x_1,\ldots,x_n) = (i_1!\ldots i_n!)a_{i_1,\ldots,i_n}$, so

$$\sum_{i_1+\ldots+i_n=p} \frac{\|P^{(i_1,\ldots,i_n)}(D_1,\ldots,D_n)Q(x_1,\ldots,x_n)\|^2}{i_1!\cdot\ldots\cdot i_n!} = \sum_{i_1+\ldots+i_n=p} \|a_{i_1,\ldots,i_n}Q(x_1,\ldots,x_n)\|^2\cdot(i_1!\cdot\ldots\cdot i_n!)$$

$$= \left(\sum_{i_1+\ldots+i_n=p}(a_{i_1,\ldots,i_n})^2\cdot(i_1!\cdot\ldots\cdot i_n!)\right)\|Q(x_1,\ldots,x_n)\|^2 = \|P\|^2\|Q\|^2 \quad .$$

Proof of C $\Rightarrow$ B: Take $R=P$ and $S=Q$.

Proof of C: Both sides are linear in P, in Q, in R, and in S, so it suffices to take them all to be typical monomials, ($P = x_1^{p_1}\cdot\ldots\cdot x_n^{p_n}$, and similarly for Q,R, and S), for which the assertion follows immediately by applying Chu's[4] identity[4]

$$\sum_{i\geq 0}\binom{r}{i}\binom{s}{p-i} = \binom{r+s}{p} \quad ,$$

to $r=r_t, s=s_t, p=p_t, (t=1\ldots n)$, (using i_t for i), and taking their product. Q.E.D.

References

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[4] Rediscovered in the 18th century by Vandermonde. Proved by counting, in two different ways, the number of ways of picking p lucky winners out of a set of r boys and s girls.