CHU’S 1303 IDENTITY IMPLIES BOMBELI’S 1990 NORM-INEQUALITY
[Via An Identity of Beauzamy and Dégot]

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Blessed are the meek: for they shall inherit the earth (Matthew V.5)

Inequalities are deep, while equalities are shallow. Nevertheless, it sometimes happens that a deep inequality, A, follows from a mere equality B, which, in turn, follows from a more general, and trivial\(^2\) identity C.

In this note we demonstrate this, following [3], with A := Bombeli’s norm inequality\(^3\), B := an identity of Reznick\([5]\), and C := an identity of Beauzamy and Dégot\([3]\). This exposition differs from the original only in the punch line: I give a 1-line proof of C, using Chu’s identity.

Let \( P(x_1, \ldots, x_n) \) and \( Q(x_1, \ldots, x_n) \) be two polynomials in \( n \) variables:

\[
P = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}, \quad Q = \sum_{i_1, \ldots, i_n \geq 0} b_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}.
\]

The Bombieri inner product\(^2\) is defined by

\[
[P, Q] := \sum_{i_1, \ldots, i_n \geq 0} (i_1! \cdots i_n!) \cdot a_{i_1, \ldots, i_n} b_{i_1, \ldots, i_n},
\]

and the Bombieri norm, by:

\[
\|P\| := \sqrt{[P, P]}.
\]

**Bombieri’s Inequality A:** Let \( P \) and \( Q \) be any homogeneous polynomials in \((x_1, \ldots, x_n)\), then

\[
\|PQ\| \geq \|P\| \|Q\|.
\]

In order to state B and C, we need to introduce the following notation. \( D_i := \frac{\partial}{\partial x_i}, \) \( (i = 1, \ldots, n) \), \( P^{(i_1, \ldots, i_n)} := D_1^{i_1} \cdots D_n^{i_n} P \), and for any polynomial \( A(x_1, \ldots, x_n) \), \( A(D_1, \ldots, D_n) \) denotes the linear partial differential operator with constant coefficients obtained by replacing \( x_i \) by \( D_i \).

\( A \) follows almost immediately from\(([5][3]):\)

**Reznick’s Identity B:** For any polynomials \( P, Q \) in \( n \) variables:

\[
\|PQ\|^2 = \sum_{i_1, \ldots, i_n \geq 0} \frac{\|P^{(i_1, \ldots, i_n)}(D_1, \ldots, D_n)Q(x_1, \ldots, x_n)\|^2}{i_1! \cdots i_n!}.
\]

**Beauzamy and Dégot’s Identity C:** For any polynomials \( P, Q, R, S \) in \( n \) variables:

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2 Trivial to verify, not to conceive!

3 It was needed by Beauzamy and Enflo in their research on deep questions on Banach spaces. It also turned out to have far reaching applications to computer algebra\([1]\).
\[ [PQ,RS] = \sum_{i_1,\ldots,i_n \geq 0} \frac{[R(i_1,\ldots,i_n)(D_1,\ldots,D_n)Q(x_1,\ldots,x_n), P(i_1,\ldots,i_n)(D_1,\ldots,D_n)S(x_1,\ldots,x_n)]}{(i_1!\ldots i_n!)} . \]

**Proof of B ⇒ A:** Pick the terms for which \( i_1 + \ldots + i_n \) equals the (total) degree of \( P \), let’s call it \( p \), and note that for those \((i_1,\ldots,i_n)\), \( P(i_1,\ldots,i_n)(x_1,\ldots,x_n) = (i_1!\ldots i_n!)a_{i_1,\ldots,i_n} \), so

\[
\sum_{i_1+\ldots+i_n=p} \frac{\|P(i_1,\ldots,i_n)(D_1,\ldots,D_n)Q(x_1,\ldots,x_n)\|^2}{i_1!\ldots i_n!} = \sum_{i_1+\ldots+i_n=p} \|a_{i_1,\ldots,i_n}Q(x_1,\ldots,x_n)\|^2 \cdot (i_1!\ldots i_n!)
\]

\[
= \left( \sum_{i_1+\ldots+i_n=p} (a_{i_1,\ldots,i_n})^2 \cdot (i_1!\ldots i_n!) \right) \|Q(x_1,\ldots,x_n)\|^2 = \|P\|^2 \|Q\|^2 .
\]

**Proof of C ⇒ B:** Take \( R = P \) and \( S = Q \).

**Proof of C:** Both sides are linear in \( P \), in \( Q \), in \( R \), and in \( S \), so it suffices to take them all to be typical monomials, \( P = x_1^{i_1} \cdot x_n^{i_n} \), and similarly for \( Q, R, \) and \( S \), for which the assertion follows immediately by applying Chu’s[4] identity

\[
\sum_{i \geq 0} \binom{r}{i} \binom{s}{p-i} = \binom{r+s}{p},
\]

to \( r = r_t, s = s_t, p = p_t, (t = 1 \ldots n), \) (using \( i_t \) for \( i \)), and taking their product. Q.E.D.

**References**


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4 Rediscovered in the 18th century by Vandermonde. Proved by counting, in two different ways, the number of ways of picking \( p \) lucky winners out of a set of \( r \) boys and \( s \) girls.