

**Alexander Burstein's Lovely Combinatorial Proof of John Noonan's Beautiful Theorem
that the number of n -permutations that contain the Pattern 321 Exactly Once Equals
 $(3/n)(2n)!/((n-3)!(n+3)!)$**

Doron ZEILBERGER¹

Alex Burstein[1] gave a lovely combinatorial proof of John Noonan's[2] lovely theorem that the number of n -permutations that contain the pattern 321 exactly once equals $\frac{3}{n} \binom{2n}{n+3}$. Burstein's proof can be made even shorter as follows. Let $C_n := (2n)!/(n!(n+1)!)$ be the Catalan numbers. It is well-known (and easy to see) that $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$. It is also well-known (and fairly easy to see) that the number of 321-avoiding n -permutations equals C_n .

Any n -permutation, π , with exactly one 321 pattern can be written as $\pi_1 c \pi_2 b \pi_3 a \pi_4$, where cba is the unique 321 pattern (so, of course $a < b < c$). All the entries to the left of b , except c , must be smaller than b , and all the entries to the right of b , except for a , must be larger than b , or else another 321 pattern would emerge. Hence $\sigma_1 := \pi_1 b \pi_2 a$ is a 321-avoiding permutation of $\{1, \dots, b\}$ that does not end with b and $\sigma_2 := c \pi_3 b \pi_4$ is a 321-avoiding permutation of $\{b, \dots, n\}$ that does not start with b . This is a bijection between the Noonan set and the set of pairs (σ_1, σ_2) as above (for some $2 \leq b \leq n-1$). For any b the number of possible σ_1 is $C_b - C_{b-1}$. Similarly, the number of possible σ_2 is $C_{n-b+1} - C_{n-b}$. Hence the desired number is

$$\begin{aligned} & \sum_{b=2}^{n-1} (C_b - C_{b-1})(C_{n-b+1} - C_{n-b}) = \sum_{b=1}^n (C_b - C_{b-1})(C_{n-b+1} - C_{n-b}) \\ &= \sum_{b=1}^n C_b C_{n-b+1} - \sum_{b=1}^n C_b C_{n-b} - \sum_{b=1}^n C_{b-1} C_{n-b+1} + \sum_{b=1}^n C_{b-1} C_{n-b} \\ &= C_{n+2} - 2C_{n+1} - 2(C_{n+1} - C_n) + C_n = C_{n+2} - 4C_{n+1} + 3C_n = \frac{3}{n} \binom{2n}{n+3} . \quad \square \end{aligned}$$

References

1. Alex Burstein, *A short proof for the number of permutations containing the pattern 321 exactly once*, Elec. J. Comb. **18(2)**(2011), #P21.
2. John Noonan, *The number of permutations containing exactly one increasing subsequence of length three*, Discrete Math. **152**(1996), 307-313 .

¹ Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. [zeilberg at math dot rutgers dot edu](mailto:zeilberg@math.rutgers.edu) , <http://www.math.rutgers.edu/~zeilberg/> . Based on a mathematics colloquium talk at Howard University, Oct. 14, 2011, 4:10-5:00pm, where Alex Burstein was presented with a \$25 check prize promised in John Noonan's 1996 Discrete Math paper. Oct 18., 2011. Exclusively published in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger <http://www.math.rutgers.edu/~zeilberg/pj.html> and arxiv.org . Supported in part by the NSF.